Math 451/551 Assignment 7

Due Tuesday, April 17

1) Let $f : X \to Y$ where X and Y are sets. Prove that

a) If $\{S_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of Y, then $f^{-1}(\cup_{\alpha\in\mathcal{I}} S_\alpha) = \cup_{\alpha\in\mathcal{I}} f^{-1}(S_\alpha)$.

b) If $\{S_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of Y, then $f^{-1}(\cap_{\alpha\in\mathcal{I}} S_\alpha) = \cap_{\alpha\in\mathcal{I}} f^{-1}(S_\alpha)$.

c) If $\{T_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of X, then $f(\cup_{\alpha\in\mathcal{I}} T_\alpha) \subseteq \cup_{\alpha\in\mathcal{I}} f(T_\alpha)$.

2) Let X and Y be metric spaces and suppose $f : X \to Y$ is continuous.

a) Prove that for all $S \subseteq Y$, $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$.

b) Provide an example of metric spaces X and Y and a continuous map $f : X \to Y$ where $\overline{f^{-1}(S)} \neq f^{-1}(\overline{S})$.

3) Let f be a continuous (in the usual metric) function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$.

a) (Exercise 4.5.7) Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

b) (Exercise 5.3.5) If, in addition, f is differentiable on $[0, 1]$ and $f'(x) \neq 1$, then f can have at most one fixed point in $[0, 1]$.

4) Let f be a twice-differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Show that if whenever

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

then $c = \dfrac{a+b}{2}$, f must be a quadratic polynomial. *Hint*: Find Steve Maguire or Azreal Dunbar.

5) Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is called *Lipschitz* if there exists an $M > 0$ such that

$$d_2(f(x), f(y)) \leq M d_1(x, y)$$

for all $x, y \in X$.

a) Show that any Lipschitz function is continuous.

b) (Exercise 5.3.1) If $f : [a, b] \to \mathbb{R}$ is differentiable and if f' is continuous on $[a, b]$, show that f is Lipschitz on $[a, b]$.