Math 451/551 Assignment 3

Due Monday, October 8

1) (Exercise 1.4.13) The Schroder-Bernstein Theorem: Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Note that if $y \in f(X)$, i.e., $y \in ran(f)$, there exists a *unique* $x \in X$ such that $f(x) = y$.

If we define $f^{-1}(y) = x$, then f^{-1} is a 1-1 function from $f(X)$ onto X. In a similar way, we can also define the 1-1 function $g^{-1} : g(Y) \to Y$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$.

a) Let $x \in X$ be arbitrary. Let the *chain* C_x be the set consisting of all elements of the form

$$\dots, f^{-1}(g^{-1}(x)), g^{-1}(x), x, f(x), g(f(x)), f(g(f(x))), \dots \tag{1}$$

Explain why the (distinct) number of elements to the left of x in the above chain may be zero, finite, or infinite.

b) Show that any two chains are either identical or completely disjoint.

c) Note that the terms of the chain in (1) alternate between elements of X and elements of Y. Given a chain C_x, we want to focus on $C_x \cap Y$, which is just the part of the chain that sits in Y.

Define the set A to be the union of all chains C_x satisfying $C_x \cap Y \subseteq f(X)$. Let B consist of the union of the remaining chains not in A. Show that any chain contained in B must be of the form

$$y, g(y), f(g(y)), g(f(g(y))), \dots$$

where y is an element of Y that is *not* in $f(X)$.

d) Let $X_1 = A \cap X$, $X_2 = B \cap X$, $Y_1 = A \cap Y$, and $Y_2 = B \cap Y$. Show that f maps X_1 *onto* Y_1 and that g maps Y_2 onto X_2. Use this information to show that X and Y have the same cardinality.

2) (continued fractions) Let $b_1 = 1$ and $b_n = 1 + \dfrac{1}{1 + b_{n-1}}$ for all $n \geq 2$. Note that $b_n \geq 1$ for all $n \in \mathbb{N}$.

a) Show that $b_{2k-1}^2 < 2$ for all $k \in \mathbb{N}$.

b) Prove that $(b_{2k-1})_{k\in\mathbb{N}}$ is increasing. Conclude that $(b_{2k+1})_{k\in\mathbb{N}}$ converges.

c) (Extra Credit) It can be shown that $(b_{2k})_{k\in\mathbb{N}}$ is decreasing and bounded below by 2, hence converges. Determine, with proof, that $(b_{2k})_{k\in\mathbb{N}}$ and $(b_{2k-1})_{k\in\mathbb{N}}$ both converge to the same number. Hence, $(b_n)_{n\in\mathbb{N}}$ converges to that number. I will accept no written solution. You must present your solution to me in my office.

3) (Exercise 2.4.6, variant) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence.

a) Prove that the sequence defined for all $n \in \mathbb{N}$ by $y_n = \sup\{a_k : k \geq n\}$ converges.

The *limit superior* of $(a_n)_{n\in\mathbb{N}}$, or $\limsup_{n\to\infty} a_n$, is then defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n.$$

where $(y_n)_{n\in\mathbb{N}}$ is the sequence from part (a) of this exercise.

b) If $n \in \mathbb{N}$ and $z_n = \inf\{a_k : k \geq n\}$, we define the *limit inferior* of $(a_n)_{n\in\mathbb{N}}$, or $\liminf_{n\to\infty}(a_n)$, by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

As in a), it can be shown that the limit exists. For the sequence

$$a_n = (-1)^n,$$

determine $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence $(a_n)_{n\in\mathbb{N}}$. Observe that part b) gives an example of a sequence that exhibits strict inequality.

d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim_{n\to\infty} a_n$ exists. In this case, all three share the same value.

Extra Credit: a) Prove that $|\mathbb{R}| = |2^{\mathbb{Z}}|$.

b) Show that if $S = \{f : \mathbb{R} \to \mathbb{R}\}$, then $|S| = |P(\mathbb{R})|$ where $P(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

Once more, I will accept no written solution for either of these problems. You must present your solution to me in my office.