Math 451/551 Assignment 4

Due Tuesday, October 16

1) (Exercise 2.7.5) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be Cauchy sequences of real numbers.

a) Show, without appealing to the Cauchy Criterion, that if $z_n = x_n + y_n$, then $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers.

b) Do the same for $w_n = x_n \cdot y_n$.

2) Recall the following construction from class: let C be the space of all Cauchy sequences of rational numbers. Let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in C$ and define an equivalence relation on C by

$$x \sim y \text{ iff } \lim_{n\to\infty} |x_n - y_n| = 0.$$

We proved that this is an equivalence relation on C. Let X be the space whose elements are equivalence classes of elements in C. If $x \in C$, let $[x]$ denote the resulting point in X.

a) Let $q \in \mathbb{Q}$ and consider the Cauchy sequence $Q \in C$,

$$Q = (q)_{n\in\mathbb{N}}.$$

Show that the map $\phi : \mathbb{Q} \to X$ given by $\phi(q) = [Q]$ is an injection.

b) Define, for elements $[x], [y] \in X$,

$$[x] \cdot [y] = [(x_n \cdot y_n)_{n\in\mathbb{N}}],$$

$$[x] + [y] = [(x_n + y_n)_{n\in\mathbb{N}}].$$

By Exercise #1, the term-wise sum or product of Cauchy sequences is again Cauchy, and so these operations define elements in X. Check that both these operations are well-defined.

c) It follows from associativity and commutativity of multiplication on rational numbers that both operations defined in b) are commutative and associative. Check that distributivity of "$\cdot$" over "$+$" holds.

3) Let (X, d) be a metric space and let Y be a nonempty subset of X. Show that d defines a metric on Y.

4) Let

$$\ell_\infty = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in \mathbb{R} \text{ and } \sup_{n\in\mathbb{N}} |a_n| < \infty\}.$$

If $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ are in ℓ_∞, define a metric on ℓ_∞ by

$$d(a, b) = \sup_{n\in\mathbb{N}} |a_n - b_n|.$$

a) Prove that d is actually a metric.

b) Deduce from problem #3 that d is a metric on the space c defined in class.

5) If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in a metric space X with metric d, show that the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges. *Hint:* For any n, m, we have by the triangle inequality

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n).$$

Now subtract $d(x_m, y_m)$ from both sides.

EXTRA CREDIT: All notation is as in Exercise #2. Again, no written solutions are permitted, but must be justified in office hours.

If $[x], [y] \in X$, define

$$D([x], [y]) = \lim_{n\to\infty} |x_n - y_n|.$$

By Exercise #5, this limit exists.

a) Show that $D([x], [y])$ is well-defined; that is, it is unchanged if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are replaced by equivalent sequences.

b) Prove that D, as given in part b), is a metric on X.

c) Prove that $\phi(\mathbb{Q})$ is dense in X, i.e., for all $\varepsilon > 0$ and $[x] \in X$, there exists a $q \in Q$ with $D([x], \phi(q)) < \varepsilon$.