Math 451/551 Assignment 5

Due Tuesday, November 13

1) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and if $k \in \mathbb{N}$, let S_k denotes the partial sums of the series $\sum_{n=1}^{\infty} a_n$. Prove that if $\lim_{k\to\infty} S_{2k} = \lim_{k\to\infty} S_{2k+1} = L$, then $\sum_{n=1}^{\infty} a_n = L$.

2) a) For all $n \in \mathbb{N}$, let $a_n \in \{0,1\}$ be any sequence of zeroes and ones. Show that $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$ always converges.

b) Let $x \in [0,1]$. Prove that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \{0,1\}$ for all $n \in \mathbb{N}$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

This is known as the *binary expansion* of x.

c) Use the result from part b) to give another proof that the set of all sequences of zeros and ones is uncountable.

3) a) Prove the *limit comparison test*: if $a_n, b_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

b) Supposing $a_n, b_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, find a counterexample to the conclusion of the limit comparison test.

4) a) Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely.

b) If $a_n \geq 0$, is it true that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges? Prove or give a counterexample.

5) Let $a_n > 0$ for all $n \in \mathbb{N}$ and suppose $\sum_{n=1}^{\infty} a_n$ diverges.

a) Show $\sum_{n=1}^{\infty} \frac{a_n}{a_n + 1}$ diverges.

b) Show $\sum_{n=1}^{\infty} \frac{a_n + 1}{a_n}$ diverges.

EXTRA CREDIT:

Prove that $\sum_{n=1}^{\infty} \frac{(2n-2)!}{4^n(n!(n-1)!)}$ converges.