Math 451/551 Assignment 6

Due Tuesday, November 20

1) Determine, with proof, which of the following sets are open, closed, or neither in $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$.

a) Any infinite interval $[a, \infty)$ for $a \in \mathbb{R}$.

b) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

c) The transcendental numbers.

2) (Exercise #3.3.4) Prove that if $K \subseteq \mathbb{R}$ is compact in the usual metric and $F \subseteq \mathbb{R}$ is closed, then $F \cap K$ is compact.

3) a) Let (X, d) be an arbitrary *complete* metric space and suppose $S \subseteq (X, d)$. Show that S is closed if and only if every Cauchy sequence in S converges to a point in S.

b) Prove that the equivalence in part a) is false in incomplete metric spaces.

4) a) Prove that the only subsets of $\mathbb{R}$ with the usual metric that are both open and closed are $\mathbb{R}$ and $\{\emptyset\}$.

b) Let X be any nonempty set and equip X with the trivial (discrete) metric d. Show that every subset of (X, d) is both open and closed.

5) (Closed-Set Version of Compactness) A collection $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ of subsets of a metric space (X, d) is said to have the *finite-intersection property* if for all $S_{i_1}, S_{i_2}, \ldots, S_{i_n} \in \mathcal{S}$, $\cap_{k=1}^{n} S_{i_k} \neq \emptyset$.

a) Let (X, d) be an arbitrary metric space. Let $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ be a collection of *closed* subsets of X. Show that X is compact if and only if for every such collection $\mathcal{S}$ with the finite intersection property,

$$\bigcap_{i \in \mathcal{I}} S_i \neq \{\emptyset\}.$$

b) Find a collection $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ of closed subsets of $\mathbb{R}$ in the usual metric satisfying the finite-intersection property, but $\cap_{i \in \mathcal{I}} S_i = \{\emptyset\}$.

EXTRA CREDIT:

Let X be any *infinite* set equipped with the trivial metric d. Show that the Heine-Borel theorem is false in (X, d).