Math 451/551 Assignment 7

Due Monday, December 10

1) Let $f : X \to Y$ where X and Y are sets. Prove that

a) If $\{S_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of Y, then $f^{-1}(\cup_{\alpha\in\mathcal{I}}S_\alpha) = \cup_{\alpha\in\mathcal{I}}f^{-1}(S_\alpha)$.

b) If $\{S_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of Y, then $f^{-1}(\cap_{\alpha\in\mathcal{I}}S_\alpha) = \cap_{\alpha\in\mathcal{I}}f^{-1}(S_\alpha)$.

c) If $\{T_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of X, then $f(\cup_{\alpha\in\mathcal{I}}T_\alpha) = \cup_{\alpha\in\mathcal{I}}f(T_\alpha)$.

2) (Exercise 3.4.9) A set S in a metric space X is called *totally disconnected* if for any $x, y \in S$, there exist separated sets A and B with $x \in A$, $y \in B$ and $S = A \cup B$. Let $C = \cap_{n=1}^{\infty} C_n$ be the Cantor set. Refer to the description given in class for the notation, or switch to the book's less precise, though more intuitive, definition.

a) Given $x, y \in C$ with $x < y$, set $\varepsilon = y - x$. For each $n \in \mathbb{N}$, C_n consists of a finite union of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval in C_N.

b) Argue that for all $x, y \in C$, there exists $z \notin C$ with $x < z < y$. Explain how this proves that there can be no interval of the form (a, b) with $a < b$ contained in C.

c) Show that C is totally disconnected.

3) a) Let f and g be two continuous functions on $\mathbb{R}$ with the usual metric and let $S \subset \mathbb{R}$ be countable. Show that if $f(x) = g(x)$ for all $x \in S^c$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

b) Let f and g be continuous functions from a metric space X into a metric space Y. Let S be a dense subset of X. Prove that $f(S)$ is dense in $f(X)$ and that if $f(s) = g(s)$ for all $s \in S$, then $f(x) = g(x)$ for all $x \in X$.

4) Characterize all continuous functions from $\mathbb{N}$ with the discrete metric into $\mathbb{R}$ with the usual metric.

5) Let f be continuous in the usual metric from the closed interval $[0, 1]$ to $[0, 1]$.

a) (Exercise 4.5.7) Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

b) (Exercise 5.3.5) If, in addition, f is differentiable on $[0, 1]$ and $f'(x) \neq 1$, then f can have at most one fixed point in $[0, 1]$.

6) Let f be a twice-differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Show that if whenever

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

then $c = \dfrac{a+b}{2}$, it must be true that f is a quadratic polynomial. *Hint*: Find Steve Maguire or Azreal Dunbar.

7) Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is called *Lipschitz* if there exists an $M > 0$ such that

$$d_2(f(x), f(y)) \leq M d_1(x, y)$$

for all $x, y \in X$.

a) Show that any Lipschitz function is continuous.

b) (Exercise 5.3.1) If $f : [a, b] \to \mathbb{R}$ is differentiable and if f' is continuous on $[a, b]$, show that f is Lipschitz on $[a, b]$.