Math 451/551 Midterm Part 1

Tuesday, October 25th

1) a) If $T \subset \mathbb{R}$, define what it means for T to be bounded above.

b) State the Completeness Axiom of the Real Numbers (or an equivalent property).

c) Give an example of a nonempty subset of $\mathbb{R}$ that is bounded above and whose supremum is equal to its infimum.

2) a) Define what it means for a set S to have the same cardinality as a set T.

b) Define what it means for a set S to be countable.

c) Give an example of a countable set that is properly contained within $\mathbb{N}$.

3) a) State the Archimedean Property of $\mathbb{R}$.

b) Define a dense subset of $\mathbb{R}$.

c) Give an example of a countable, dense subset of $\mathbb{R}$.

4) a) Define a convergent sequence of real numbers.

b) Define a bounded sequence of real numbers.

c) State the Monotone Convergence Theorem.

d) Give an example of a bounded sequence of real numbers that does not converge.

5) a) Define a metric on a set X.

b) State what it means for a metric space (X, d) to be complete.

c) Give an example of a complete metric space.

6) Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence of real numbers. Prove that $(cx_n)_{n=1}^{\infty}$ is Cauchy for all real numbers $c \neq 0$.

Math 451/551 Midterm Part 2

Thursday, October 27th

1) Prove that for all real numbers x and y, $|x - y| \leq |x| + |y|$.

2) Let
$$\mathcal{S} = \{T \subset \mathbb{N} \mid |T| < \infty\}.$$
Prove that $\mathcal{S}$ is countable.

3) Do ONE of the following two questions. If you do both, I will grade the problem you do WORSE on.

a) Let $(x_n)_{n=1}^{\infty}$ be the sequence

$$x_n = 1 - \sum_{i=1}^{n} \frac{1}{n+i}.$$

Prove that $(x_n)_{n=1}^{\infty}$ converges.

-OR-

b) Let S be a countable, dense subset of $\mathbb{R}$. Prove that S^c is also a dense subset of $\mathbb{R}$.