Math 412/512 Final

Wednesday, April 25th

The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem, with the possible exception of #1. Use them wisely.

1. a) (5 points) Give examples of two dense, proper subsets of $\mathbb{R}$ with the usual metric.

   b) (8 points) Define a Cauchy sequence of real numbers and explain what is meant by completeness of the real numbers.

2. (25 points) Prove that the rational numbers are dense in the real numbers.

3. a) (7 points) State the comparison test for series of real numbers.

   b) (6 points) State the Monotone Convergence Theorem for sequences of real numbers.

4. a) (10 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $a_n \geq 0$ for all $n \in \mathbb{N}$ and suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that if $(b_n)_{n\in\mathbb{N}}$ is any bounded sequence of real numbers, then $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

b) (15 points) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Suppose $0 < x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} x_n = x$. Prove that $y_k = (x_1 x_2 \cdots x_k)^{1/k}$ converges. (*10 points extra credit*) Show $\lim_{k\to\infty} y_k = x$.

5. a) (6 points) Define a connected subset of a metric space (X, d).

   b) (6 points) Define what it means for $f : \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $a \in \mathbb{R}$.

6. a) (12 points) Consider $\mathbb{N}$ as a metric space with the discrete metric

$$d(m,n) = \begin{cases} 0, & n = m \\ 1, & n \neq m \end{cases}$$

for all $m, n \in \mathbb{N}$. Characterize the connected subsets of $(\mathbb{N}, d)$, with proof.

b) (13 points) Let $f : \mathbb{R} \to \mathbb{R}$. Let L be a real number and suppose

$$\lim_{x \to 0} \frac{f(2x) - f(x)}{x} = L.$$

Assuming $f'(0)$ exists, show that $f'(0) = L$.

7. a) (6 points) State the Bolzano-Weierstrass Theorem.

   b) (6 points) State the Extreme Value Theorem for a continuous, real-valued function f from a metric space (X, d) to $\mathbb{R}$ with the usual metric.

8. Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) (25 points) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous in the usual metric. Let $x \in \mathbb{R}$ and define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = x$ and $x_n = f(x_{n-1})$ for all $n \geq 2$. Supposing that $(x_n)_{n\in\mathbb{N}}$ is bounded, prove that there exists a $y \in \mathbb{R}$ with $f(y) = y$.

-OR-

b) (25 points) Let K be a nonempty, compact subset of a metric space X and let $x \in X$. Define

$$d(x, K) = \inf_{y\in K} d(x, y).$$

Prove that there exists a z in K with $d(x, z) = d(x, K)$, i.e., the infimum is attained and so is a minimum.