Math 412/512 Midterm

IN-CLASS PORTION

Tuesday, March 13th

1) a) Define what it means for two sets S and T to have the same cardinality.

b) Define what it means for a set S to be countable.

c) Give an example of 2 infinite sets that do not have the same cardinality.

2) a) State the Completeness Axiom for the real numbers $\mathbb{R}$ (equivalent statements are acceptable).

b) We have shown that the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$. Explain what this means using mathematical terminology.

c) Give an example of a dense, infinite, proper subset S of $\mathbb{R}$ that is NOT $\mathbb{Q}$.

3) a) Define what it means for a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ to converge to a real number L.

b) State the Monotone Convergence Theorem.

c) Give an example of a bounded sequence of real numbers that does not converge.

4) a) Define a subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers.

b) State the Bolzano-Weierstrass Theorem.

c) Give an example of a sequence of real numbers with no convergent subsequence.

5) a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define what it means for the series $\sum_{n=1}^{\infty} x_n$ to converge.

b) State the comparison test.

c) Give an example of a convergent series with all nonnegative terms.

7) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that converges to zero. Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. If $z_n = x_n \cdot y_n$ for all $n \in \mathbb{N}$, prove that $(z_n)_{n\in\mathbb{N}}$ converges to zero.

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TAKE-HOME PORTION

Due Thursday, March 15th

General rules: You may use your textbook and notes as resources, but no other texts and certainly no people other than yourself.

1) Let $a_1 = 2$ and define a_n recursively for $n \in \mathbb{N}$, $n \geq 2$, by

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{3}{a_{n-1}}\right).$$

a) Prove that $a_n^2 \geq 3$ for all $n \in \mathbb{N}$. *Hint:* $\frac{x+y}{2} \geq \sqrt{xy}$ for all $x, y \geq 0$.

b) Prove that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

c) Deduce that $(a_n)_{n\in\mathbb{N}}$ converges and prove that $\lim\limits_{n\to\infty} a_n = \sqrt{3}$.

2) Consider $\mathbb{R}^2$ as a metric space with the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

a) Prove that $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$.

b) Prove that there exists a subset S of $\mathbb{R}^2$ such that S and S^c are both uncountable and S is dense in $\mathbb{R}^2$.

3) a) For all $n \in \mathbb{N}$, let $a_n \in \{0, 1\}$ be any sequence of zeroes and ones. Show that $\sum\limits_{n=1}^{\infty} \frac{a_n}{2^n}$ always converges.

b) Let $x \in [0, 1]$. Prove that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \{0, 1\}$ for all $n \in \mathbb{N}$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

This is known as the *binary expansion* of x.

4) Use the Monotone Convergence Theorem (every bounded, monotonic sequence of real numbers converges) to give a proof of the Nested Interval Property (the intersection of any nested sequence of closed intervals is nonempty).