Math 451/551 Final

Wednesday, April 22nd

1) a) Define a rational number $x \in \mathbb{R}$.

b) Define a dense subset S of $(\mathbb{R}, |\cdot|)$.

2) a) If $x \in \mathbb{R} \backslash \mathbb{Q}$, show that $1/x \in \mathbb{R} \backslash \mathbb{Q}$.

b) Let $S \subseteq \mathbb{R}$ and suppose S^c is countable. Show that S is dense in $\mathbb{R}$.

3) a) Define an open set O in a metric space (X, d).

b) In a metric space (X, d), define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to converge to $x \in X$.

4) Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that $\lim_{n\to\infty} x_n = x$ if and only if every open set containing x contains all but finitely many points of $(x_n)_{n=1}^{\infty}$.

5) a) State the comparison test for series of nonnegative real numbers.

b) State the Mean Value Theorem for $f : [a, b] \to \mathbb{R}$.

6) a) Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of nonnegative real numbers. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{(n+1)^3}$$

converges.

b) Let f be continuous in the usual metric from the closed interval $[0,1]$ to $[0,1]$ and suppose f is differentiable on $(0,1)$ and $f'(x) \neq 1$. Prove that there can be at most one point $x \in [0,1]$ with $f(x) = x$.

7) a) State the Intermediate Value Theorem for $f : [a, b] \to \mathbb{R}$.

b) Define what it means for a subset K of a metric space (X, d) to be sequentially compact.

8) Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let f be a real-valued continuous function on $[0,1]$ and suppose $f(0) = f(1)$. Prove that there exists a point $x \in [0, 1/2]$ such that $f(x) = f(x+1/2)$.

-OR-

b) Let (X, d) be a metric space and let $K \subseteq X$ be compact. Prove that K is closed.