Math 451/551 Midterm Part 1

Tuesday, March 17th

1) a) Define $\mathbb{Z}$ via an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

b) Define $\mathbb{Q}$ via an equivalence relation on $\mathbb{Z} \times \mathbb{Z} \backslash \{0\}$.

2) a) Define what it means for a set S to have the same cardinality as a set T.

b) Define what it means for a set S to be uncountable.

c) Give an example of an uncountable set that is not equal to $\mathbb{R}$.

3) a) Define what it means for a subset S of $\mathbb{R}$ to be bounded.

b) Define the infimum of a bounded set S of real numbers.

c) Give an example of a bounded subset S of $\mathbb{R}$ whose infimum is not in S, and provide the infimum.

4) a) Define a Cauchy sequence of real numbers.

b) Without proof, state whether every Cauchy sequence converges for the following sets: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

c) Define $\mathbb{R}$ via an equivalence relation on the space $\mathcal{X}$, of Cauchy sequences of rational numbers.

5) a) Define what it means for $\sum_{n=1}^{\infty} a_n$ to converge absolutely, where $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers.

b) State the Cauchy Condensation Test.

c) Give an example of a series that converges but does not converge absolutely, i.e., a series that converges conditionally.

6) Let x and y be real numbers and suppose $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$. Prove that $xy \in \mathbb{Q}$ if and only if $x = 0$.

Math 412/512 Midterm Part 2

Thursday, March 19th

1) Suppose that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} |a_n| = |L|$.

2) Prove that the irrational numbers are dense in $\mathbb{R}$.

3) Do ONE of the following two questions. If you do both, I will grade the problem you do WORSE on.

a) Suppose $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

-OR-

b) Let $\mathcal{S} = \{f : \mathbb{R} \to \mathbb{R}\}$, i.e., $\mathcal{S}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $\mathcal{S}$ is uncountable.