Math 451/551 Final

Monday, December 17th

The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem, with the exception of 1b). Use them wisely.

1. a) (7 points) If (X, d_1) and (Y, d_2) are metric spaces, define what it means for a function $f : (X, d_1) \to (Y, d_2)$ to be continuous at a point. Equivalent statements are acceptable.

   b) (6 points) Give an example of a function $f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ that is discontinuous at every point.

2. (15 points) Let (X, d) be a metric space and let $f : (X, d) \to (\mathbb{R}, |\cdot|)$ be continuous. Let

$$\mathcal{Z}(f) = \{x \in X \mid f(x) = 0\}.$$

Prove that $\mathcal{Z}(f)$ is closed.

3. a) (5 points) State the monotone convergence theorem.

   b) (8 points) State the nested interval property.

4. a) (15 points) Let $x_1 = \dfrac{1}{4}$ and for $n > 1$, define $x_n = (x_{n-1})^{\frac{1}{4}}$. Prove that $(x_n)_{n=1}^{\infty}$ converges.

   b) (15 points) Prove that the monotone convergence theorem implies the nested interval property.

5. a) (6 points) If $f : \mathbb{R} \to \mathbb{R}$, define what it means for f to be differentiable at a point $a \in \mathbb{R}$. Equivalent statements are acceptable.

   b) (6 points) State the comparison test.

6. a) (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| < (x - y)^2.$$

Show that there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in \mathbb{R}$, i.e., f is constant.

b) (15 points) Show that if $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} (n^2 a_n)$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.

7. a) (6 points) State the intermediate value theorem.

   b) (6 points) If S is a subset of a metric space (X, d), define what it means for S to be compact. Equivalent statements are acceptable.

8. Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) (25 points) Let $f : [a, b] \to \mathbb{R}$ be continuous in the usual metric on both its domain and range. Suppose that for any point $x \in [a, b]$, there is another point $x' \in [a, b]$ such that $|f(x')| \leq |f(x)|/2$. Prove that there exists a point $x_0 \in [a, b]$ where $f(x_0) = 0$.

-OR-

b) (25 points) Let (X, d) be a compact metric space and let $f : X \to Y \subset X$ satisfy

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. Prove that $Y = X$.