Math 412/512 Midterm

IN-CLASS PORTION

Monday, October 29th

1) a) Define an injection between two sets S and T.

b) Define what it means for a set S to be uncountably infinite.

c) Give an example of 2 infinite sets that have the same cardinality.

2) a) Define the supremum of a bounded set of real numbers.

b) State the Archimedean Property of the real numbers.

c) Give an example of a bounded set of real numbers whose supremum is an integer.

3) a) State the Nested Interval Property.

b) Give an example of a nested sequence of intervals whose intersection is empty.

4) a) State the Monotone Convergence Theorem.

b) State the Bolzano-Weierstrass Theorem.

c) Give an example of a sequence of real numbers that contains a monotonically decreasing convergent subsequence.

5) a) Provide the definition of a metric space.

b) Define a Cauchy sequence in a metric space.

c) Define what it means for a metric space (X, d) to be complete.

d) Give an example of a metric space that is complete.

6) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and suppose $\lim_{n\to\infty} x_n = L \neq 0$. Let $y_n = (-1)^n x_n$. Prove that $(y_n)_{n=1}^{\infty}$ diverges.

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TAKE-HOME PORTION

Due Friday, November 2

GENERAL RULES: You may use your textbook and notes as resources, but no other texts and certainly NO PEOPLE other than yourself.

1) Define $a_1 = \sqrt{5}$ and $a_{n+1} = \sqrt{4 + \sqrt{a_n}}$ for all $n \in \mathbb{N}$.

a) Prove that $(a_n)_{n=1}^{\infty}$ is bounded above by 3.

b) Prove that $(a_n)_{n=1}^{\infty}$ is monotonically increasing. Conclude $(a_n)_{n=1}^{\infty}$ converges

2) Let S and T be bounded sets of negative real numbers. Define

$$ST = \{st \mid s \in S, t \in T\}.$$

Show that ST is bounded and that $\inf(ST) = \sup(S) \cdot \sup(T)$.

3) a) Consider $\mathbb{Z}$ with the trivial (discrete) metric

$$d(n, m) = \begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases}$$

for all $n, m \in \mathbb{Z}$. Prove that $(\mathbb{Z}, d)$ is complete.

b) Find an example of a metric space (Y, d) with the following property: Y contains a subset X with $|X| > 1$ such that (X, d) is a complete metric space, yet (Y, d) is NOT a complete metric space. Be sure to show that your example satisfies all the given hypotheses.

4) Prove that $\mathbb{R}$ contains uncountably many distinct countable dense subsets.