Math 451/551 Final

Friday, December 16th

The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem, with the exception of 1b). Use them wisely.

1. a) Define a metric d on a set X. Alternatively, define the metric space (X, d).

   b) State the Heine Borel Theorem.

2. a) For $x, y \in \mathbb{R}$, define

$$f(x, y) = \frac{|x - y|}{|y| + 1}.$$

Prove that f is NOT a metric on $\mathbb{R}$.

b) Let $S, T \subset \mathbb{R}$ be compact. Show that $S \cup T$ is also compact.

3. Define a dense subset S of the real numbers $\mathbb{R}$.

4. Prove that $\mathbb{R}$ has infinitely many dense subsets.

5. a) If $f : \mathbb{R} \to \mathbb{R}$, define what it means for f to be differentiable at a point $a \in \mathbb{R}$. Equivalent statements are acceptable.

   b) Define an infinitesimal hyperreal number.

6. a) Prove the power rule: $(x^n)' = nx^{n-1}$ for $n \geq 1$ and $x \in \mathbb{R}$.

   b) Show that if $\varepsilon, \delta \in {}^*\mathbb{R}$ are infinitesimal, then $\varepsilon + \delta$ is also infinitesimal.

7. a) State the comparison test for infinite series of real numbers with non-negative terms.

   b) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. State the Bolzano-Weierstrass Theorem.

8. Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers and suppose $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge. Show that

$$\sum_{n=1}^{\infty} |a_n b_n|$$

converges.

-OR-

b) Prove that every sequence $(x_n)_{n=1}^{\infty}$ of real numbers has a subsequence that is either monotonically increasing or monotonically decreasing.