Math 451/551 Final

Tuesday, December 14th

The odd-numbered problems are definitions and theorems meant to aid you in the subsequent even-numbered problem. Use them wisely.

1. a) State the Principle of Mathematical Induction.

   b) Define what it means for a subset S of $\mathbb{R}$ to be dense in $\mathbb{R}$.

2. a) Let T be a nonempty subset of $\mathbb{N}$ and suppose $|T| < \infty$. Prove that there exists an $n \in T$ with $n \geq k$ for all $k \in T$.

   b) Prove that $S = \{x + \sqrt{2} \mid x \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.

3. a) Define a Cauchy sequence in $\mathbb{R}$.

b) State the comparison test for series of real numbers with non-negative terms.

4. a) Prove that if (a_n) and (b_n) are Cauchy sequences of real numbers, then $(a_n + b_n)$ is a Cauchy sequence of real numbers.

b) Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$ and that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges.

5. a) State the Heine-Borel Theorem.

   b) Define a connected subset S of a metric space (X, d).

6. a) Determine whether $\mathbb{Q} \cap [0, 1]$ is a compact subset of $(\mathbb{R}, |\cdot|)$.

   b) Prove that the intersection of two connected subsets S and T of $(\mathbb{R}, |\cdot|)$ is connected.

7. a) Give the sequential definition of compactness for a subset S of a metric space (X, d).

   b) State the Mean Value Theorem.

8. Do ONE of the following two problems. If you attempt both, I will grade the problem you do WORSE on.

a) Let $C([0,1])$ denote the collection of all *continuous* functions from $[0,1]$ to $\mathbb{R}$ and define, for $f, g \in C([0,1])$,

$$d(f,g) = \max_{x \in [0,1]} \{|f(x) - g(x)|\}.$$

Then d is a metric on $C([0,1])$ (you do not have to prove this). Show that $B(0,1)$ is not compact, where "0" is the function that is constantly zero.

-OR-

b) Let f be a continuous, real-valued function on $[0,1]$. Suppose f is differentiable on $(0,1)$ $f(0) = 0$, and that $|f'(x)| < 1$ for all $x \in (0,1)$. Prove that $f(x) \leq x$ for all $x \in (0,1)$.