Constructing the Real Numbers

Let $(x_n)_{n=1}^{\infty}$ be a sequence of rational numbers. We say $(x_n)_{n=1}^{\infty}$ is Cauchy in $\mathbb{Q}$ if $\forall\ q \in \mathbb{Q},\ q > 0,\ \exists\ N \in \mathbb{N},\ |x_n - x_m| < q$ $\forall\ n, m \geq N$.

Similarly, we say

$$\lim_{n\to\infty} x_n = L \in \mathbb{Q} \text{ if}$$

$$\forall\ q \in \mathbb{Q},\ q > 0,\ \exists\ N \in \mathbb{N},$$

$$|x_n - L| < q$$

$$\forall\ n \geq N.$$

Finally, $(x_n)_{n=1}^{\infty}$ is bounded above if $\exists\, M \in \mathbb{Q}$, $x_n \leq M \ \forall\, n \in \mathbb{N}$ and bounded below if $\exists\, L \in \mathbb{Q}$, $L \leq x_n \ \forall\, n \in \mathbb{N}$.

We say $(x_n)_{n=1}^{\infty}$ is bounded if $(x_n)_{n=1}^{\infty}$ is bounded above and below.

Lemma: Every Cauchy sequence (of reals or rationals) is bounded.

proof: Let $(x_n)_{n=1}^{\infty}$ be Cauchy. Then $\exists\, N \in \mathbb{N}$,

$$|x_n - x_m| < 1 \quad \forall\, n, m \geq N.$$

Hence

$$\begin{aligned} 1 &> |x_n - x_N| \\ &> \big|\, |x_n| - |x_N| \,\big| \\ &> |x_n| - |x_N| \end{aligned}$$

$\forall\; n \geq N.$

Hence, $\forall\, n \geq N$,

$$|x_n| < |x_N| + 1.$$

Now let

$$L = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N|+1\}$$

Then $|x_n| \leq L \;\; \forall\, n \in \mathbb{N}$

$\Rightarrow (x_n)_{n=1}^{\infty}$ is bounded. $\square$

Now let $\overline{X}_0$ be the space of all Cauchy sequences of rational numbers. For $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty} \in \overline{X}_0$, define

$$x + y = (x_n + y_n)_{n=1}^{\infty}$$

$$x \cdot y = (x_n y_n)_{n=1}^{\infty}$$

It follows from associativity and commutativity of addition and multiplication in $\mathbb{Q}$ that "+" and "$\cdot$" are associative and commutative.

Also, distributivity of "$\cdot$" over "+" follows from distributivity of a multiplication over addition in $\mathbb{Q}$.

Proposition: If $x, y \in \overline{X}_0$, then $x+y, x \cdot y \in \overline{X}_0$.

proof The proof is much like that of sums and products of convergent sequences.

For $x+y$, $\forall q \in \mathbb{Q}, q > 0$, $\exists N_1, N_2 \in \mathbb{N}$,

$|x_n - x_m| < \frac{q}{2} \quad \forall n, m \geq N_1$

$|y_\ell - y_k| < \frac{q}{2} \quad \forall \ell, k \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Then

$\forall\ n, m \geq N$,

$$|x_n + y_n - x_m - y_m|$$

$$\leq |x_n - x_m| + |y_n - y_m|$$

$$< \frac{a}{2} + \frac{a}{2} = a$$

$\Rightarrow (x_n + y_n)_{n=1}^{\infty}$ is

Cauchy.

Now define an equivalence relation "$\sim$" on $\mathbb{X}_0$ by

$$\boxed{(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \text{ if } \lim_{n\to\infty} |x_n - y_n| = 0}$$

We need to show this is indeed an equivalence relation.

1) <u>Reflexivity</u> $(x_n)_{n=1}^{\infty} \sim (x_n)_{n=1}^{\infty}$

$$\lim_{n\to\infty} |x_n - x_n| = \lim_{n\to\infty} 0$$

$$= 0$$

2) <u>Symmetry</u> If $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$,

then $(y_n)_{n=1}^{\infty} \sim (x_n)_{n=1}^{\infty}$.

$$\lim_{n\to\infty} |y_n - x_n| = \lim_{n\to\infty} |x_n - y_n|$$

$$= 0$$

3) <u>Transitivity</u> If $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty} \sim (z_n)_{n=1}^{\infty}$, then $(x_n)_{n=1}^{\infty} \sim (z_n)_{n=1}^{\infty}$.

$$0 \leq |x_n - z_n| = |x_n - y_n + y_n - z_n|$$

$$\leq |x_n - y_n| + |y_n - z_n|$$

Since $\lim_{n\to\infty}\left(|x_n - y_n| + |y_n - z_n|\right)$

$= 0 + 0 = 0$,

$\lim_{n\to\infty} |x_n - z_n| = 0$ ✓

(Squeeze theorem)

Hence, "$\sim$" is an equivalence relation on $\mathbb{X}_0$.

Set $\mathbb{X} = \mathbb{X}_0/_\sim$.

The goal is to convince ourselves that $\mathbb{X} = \mathbb{R}$.

To that end, define "+" and "." on $\mathbb{X}$ by,

if $x, y \in \mathbb{X}_0$,

$$[x] + [y] = [x+y]$$

and

$$[x] \cdot [y] = [xy]$$

Proposition: "$+$" and "$\cdot$" are well-defined.

proof: Let $[x],[y] \in \mathbb{X}$ and suppose $x=(x_n)_{n=1}^{\infty}$, $y=(y_n)_{n=1}^{\infty}$. Let

$$x \sim x' = (x'_n)_{n=1}^{\infty},$$

$$y \sim y' = (y'_n)_{n=1}^{\infty}.$$

Then

$$0 \le |x_n + y_n - x_n' - y_n'|$$

$$\le |x_n - x_n'| + |y_n - y_n'|,$$

and $\lim_{n\to\infty} |x_n - x_n'| = \lim_{n\to\infty} |y_n - y_n'|$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} |x_n + y_n - x_n' - y_n'| = 0,$$

so $[x] + [y] = [x'] + [y']$.

Now

$$0 \leq |x_n y_n - x_n' y_n'|$$

$$= |x_n y_n - x_n y_n' + x_n y_n' - x_n' y_n'|$$

$$\leq |x_n y_n - x_n y_n'| + |x_n y_n' - x_n' y_n'|$$

$$= |x_n||y_n - y_n'| + |y_n'||x_n - x_n'|.$$

Since $(x_n)_{n=1}^{\infty}$ and $(y_n')_{n=1}^{\infty}$ are Cauchy, they are bounded, and so $\exists\ M_1, M_2 \in \mathbb{Q}$, $|x_n| \leq M_1$, $|y_n'| \leq M_2$ $\forall\ n \in \mathbb{N}$.

Hence

$$|x_n||y_n - y_n'| + |y_n'||x_n - x_n'|$$

$$\leq M_1|y_n - y_n'| + M_2|x_n - x_n'|,$$

and since

$$\lim_{n\to\infty} |y_n - y_n'| = \lim_{n\to\infty} |x_n - x_n'| = 0,$$

$$\lim_{n\to\infty} |x_n y_n - x_n' y_n'| = 0$$

$$\Rightarrow [x]\cdot[y] = [x']\cdot[y'].$$

Now for products, we know $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are bounded, so $\exists\, M_1, M_2 \in \mathbb{Q}$,

$|x_n| \leq M_1$ and $|y_n| \leq M_2$

$\forall\, n \in \mathbb{N}$. We have

$$
\begin{aligned}
&|x_n y_n - x_m y_m| \\
&\leq |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\
&\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \\
&= |x_n|\,|y_n - y_m| + |y_m|\,|x_n - x_m| \\
&\leq M_1 |y_n - y_m| + |M_2|\,|x_n - x_m|
\end{aligned}
$$

Now let $q \in \mathbb{Q}$, $q > 0$, and choose $N_1, N_2 \in \mathbb{N}$

$$|x_n - x_m| < \frac{q}{2M_2} \quad \forall\, n, m \geq N_1$$

$$|y_l - y_k| < \frac{q}{2M_1} \quad \forall\, k, l \geq N_2.$$

Then with $N = \max\{N_1, N_2\}$,

$\forall\ n, m \geq N$,

$$
\begin{aligned}
&|x_n y_n - x_m y_m| \\
&\leq |x_n||y_n - y_m| + |y_m||x_n - x_m| \\
&< M_1 |y_n - y_m| + M_2 |x_n - x_m| \\
&< M_1\left(\frac{\varepsilon}{2M_1}\right) + M_2\left(\frac{\varepsilon}{2M_2}\right) \\
&= \varepsilon \implies (x_n y_n)_{n=1}^{\infty} \text{ is}
\end{aligned}
$$

Cauchy. $\square$

Define $\varphi : \mathbb{Q} \to \overline{X}$,

$$\varphi(q) = [(q)_{n=1}^{\infty}],$$

where $(q)_{n=1}^{\infty}$ is the sequence with a q in every entry. It is trivial that $(q)_{n=1}^{\infty}$ is a Cauchy sequence of rationals, and it is easy to check that φ is injective.

Proposition: $\forall\ [x] \in \mathbb{X}$,

$x = (x_n)_{n=1}^{\infty}$,

1) $\varphi(0) + [x] = [x]$

2) $\varphi(1) \cdot [x] = [x]$

3) If $-x = (-x_n)_{n=1}^{\infty}$,

$$[x] + [-x] = \varphi(0)$$

4) If $[x] \neq \varphi(0)$, $\exists\ [z] \in \mathbb{X}$,

$$[x][z] = \varphi(1)$$

proof:

1) $[x] + \varphi(0)$

$= \left[(x_n + 0)_{n=1}^{\infty} \right]$

$= \left[(x_n)_{n=1}^{\infty} \right] = [x]$ ✓

2) $[x] \cdot \varphi(1)$

$= \left[(x_n \cdot 1)_{n=1}^{\infty} \right]$

$= \left[(x_n)_{n=1}^{\infty} \right]$

$= [x]$ ✓

3) $[x] + [-x]$

$$= \left[(x_n - x_n)_{n=1}^{\infty} \right]$$

$$= \left[(0)_{n=1}^{\infty} \right] = \varphi(0)$$ ✓

4) Since $[x] \neq \varphi(0)$, $\exists$

$N_1 \in \mathbb{N}$, $q \in \mathbb{Q}$,

$$|x_n| > q > 0$$

$\forall \; n \geq N_1$.

We define $z_n = \begin{cases} 1, & n < N_1 \\ \frac{1}{x_n}, & n \geq N_1. \end{cases}$

Then $\forall\ n \geq N_1$,

$$x_n \cdot z_n = x_n \cdot \frac{1}{x_n}$$

$$= 1$$

$$\Rightarrow \left[(x_n z_n)_{n=1}^{\infty} \right] = \varphi(1)$$

It remains to show $(z_n)_{n=1}^{\infty}$ is actually a Cauchy sequence!

Let $p \in \mathbb{Q}$, $p > 0$. Since $(x_n)_{n=1}^{\infty}$ is Cauchy, $\exists\ N_2 \in \mathbb{N}$,

$$|x_n - x_m| < q^2 p$$

$\forall\ n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

Then $\forall\, n \geq N$,

$$|z_n - z_m| = \left|\frac{1}{x_n} - \frac{1}{x_m}\right|$$

$$= \left|\frac{x_m - x_n}{x_n x_m}\right|$$

$$< \frac{1}{q^2}|x_m - x_n|$$

$$< \frac{p\,\varepsilon^2}{q^2} = p$$

$\Rightarrow (z_n)_{n=1}^{\infty}$ is Cauchy $\square$

The fact that "+" and "$\cdot$" are commutative and associative on $\mathbb{X}$ follows from commutativity and associativity of "+" and "$\cdot$" on $\mathbb{X}_0$, as does distributivity of "$\cdot$" over "+". We will now develop a total order on $\mathbb{X}$.

If $[x], [y] \in \mathbb{X}$,

$x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$,

we define

$\boxed{[x] < [y]}$ if $\exists$

$q \in \mathbb{Q}$, $q > 0$, and $N \in \mathbb{N}$

such that

$\boxed{y_n - x_n > q}$ $\forall n \geq N$

We say $[x] \leq [y]$ if either

$[x] < [y]$ or $[x] = [y]$

Proposition: "$\leq$" is well-defined.

proof: Suppose $(x_n)_{n=1}^{\infty} \sim (x_n')_{n=1}^{\infty}$

and $(y_n)_{n=1}^{\infty} \sim (y_n')_{n=1}^{\infty}$.

If $[x]=[y]$, then by transitivity of the equivalence relation, $[(x_n')_{n=1}^{\infty}] = [x]$

$$= [y]$$

$$= [(y_n')_{n=1}^{\infty}]$$

Assume, then, that $[x] < [y]$.

Hence, $\exists$ $q \in \mathbb{Q}$, $N_1 \in \mathbb{N}$,

$$y_n - x_n > q > 0 \quad \forall \ n \geq N_1.$$

Now by the definition of "$\sim$",

$\exists$ $N_2, N_3 \in \mathbb{N}$

$$|x_n - x_n'| < \frac{q}{4}$$

$$|y_m - y_m'| < \frac{q}{4}$$

$$\forall \quad n \geq N_2, \ m \geq N_3.$$

Now let $N = \max\{N_1, N_2, N_3\}$.

Then $\forall\ n \geq N$,

$$y_n' - x_n' = y_n' - y_n + y_n - x_n + x_n - x_n'$$

$$> -\frac{q}{4} + y_n - x_n - \frac{q}{4}$$

$$> -\frac{q}{4} + q - \frac{q}{4}$$

$$= \frac{q}{2} > 0$$

$\Rightarrow$ " $\leq$ " is well-defined.

$\square$

Theorem: "$\leq$" is a total order on $\mathbb{X}$.

Proof: We first show "$\leq$" is a partial order.

Let $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$, and $z = (z_n)_{n=1}^{\infty} \in \mathbb{X}_0$.

1) Reflexivity since $x \sim x$,

$$[x] \leq [x]$$

2) <u>Symmetry</u> Suppose $[x] \leq [y]$

and $[y] \leq [x]$.

If $[x] < [y]$, then $\exists\, q \in \mathbb{Q}$,

$N \in \mathbb{N}$, $y_n - x_n > q > 0$

$\Rightarrow [y] < [x]$ is

impossible. Hence,

$[x] = [y]$.

3) Transitivity Suppose

$[x] \leq [y]$, $[y] \leq [z]$.

If either $[x] = [y]$ or

$[y] = [z]$, it is trivial

that $[x] \leq [z]$, so assume

$[x] < [y]$, $[y] < [z]$.

Then $\exists$ $p, q \in \mathbb{Q}$, $N_1, N_2 \in \mathbb{N}$

such that

$$z_n - y_n > p > 0$$

$$y_m - x_m > q > 0$$

$\forall$ $n \geq N_1$, $m \geq N_2$.

Let $N = \max\{N_1, N_2\}$.

Then $\forall\ n \geq N$,

$$z_n - x_n = z_n - y_n + y_n - x_n$$
$$> p + q > 0$$

We have shown that "$\leq$" is a partial order on $\mathbb{X}$.

We now must show, $\forall$ $[x], [y] \in \mathbb{X}$, either

$$[x] \leq [y] \text{ or }$$

$$[y] \leq [x].$$

Suppose that $[x] < [y]$

is not true. Then

$\forall\ q \in \mathbb{Q},\ q > 0$ and $N \in \mathbb{N}$

$\exists\ n > N,$

$y_n - x_n \leq q.$

Hence, $\forall\ k, N \in \mathbb{N},\ \exists\ n > N,$

$y_n - x_n \leq \frac{1}{k}.$

Since $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$

are Cauchy, $\exists\, N_1, N_2 \in \mathbb{N}$,

$$|x_m - x_e| < \frac{1}{4k}$$

and $|y_s - y_t| < \dfrac{1}{4k}$.

Let $N = \max\{N_1, N_2\}$.

Then $\exists\, n > N$,

$$y_n - x_n \leq \frac{1}{2k}$$

Then $\forall\ m \geq n$,

$$
\begin{aligned}
y_m - x_m &= y_m - y_n + y_n - x_n + x_n - x_m \\
&< \frac{1}{4k} + y_n - x_n + \frac{1}{4k} \\
&\leq \frac{1}{4k} + \frac{1}{2k} + \frac{1}{4k} \\
&= \frac{1}{k}
\end{aligned}
$$

Now if $[y] < [x]$ is also not true, then by the same reasoning, $\exists\ r \in \mathbb{N}$ s.t. $\forall\ m \geq r$, $x_m - y_m < \frac{1}{k}$.

Then with $M = \max\{r, n\}$,

$$|x_m - y_m| < \frac{1}{k}$$

$\forall\, m \geq M \Rightarrow [x] = [y]$.

Hence, either $[x] > [y]$, $[x] < y$, or $[x] = [y]$

$\Rightarrow [x] \leq [y]$ or $[y] \leq [x]$. $\square$

Proposition: Let $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$, $z = (z_n)_{n=1}^{\infty} \in X_0$.

Then

1) If $[x] \leq [y]$,

$$[x] + [z] \leq [y] + [z]$$

2) If $[x], [y] \geq \varphi(0)$, then

$$[x] \cdot [y] \geq \varphi(0).$$

proof: 1) If $[x]=[y]$, then

$$[x]+[z]=[y]+[z],$$

so suppose $[x]<[y]$.

Then $\exists\ q\in\mathbb{Q},\ N\in\mathbb{N}$

$$y_n-x_n>q>0.$$

Then

$$(y_n+z_n)-(x_n+z_n)$$

$$=y_n-x_n>q>0$$

Let $N = \max\{N_1, N_2\}$. Then

$\forall\ n \geq N,$

$$x_n \cdot y_n = (-x_n)(-y_n)$$

$$> pq > 0$$

$$\Rightarrow \quad [x]\cdot[y] > \varphi(0).$$

$\square$

2) If either $[x] = \varphi(0)$ or $[y] = \varphi(0)$, then $[x]\cdot[y] = \varphi(0)$, so

suppose $[x], [y] < \varphi(0)$. Then

$\exists\ p, q \in \mathbb{Q},\ N_1, N_2 \in \mathbb{N},$

$$-x_n > q > 0$$

$$-y_m > p > 0$$

$\forall\ n \geq N_1,\ m \geq N_2.$

Remark: One can also similarly show

1) If $[x], [y] > \varphi(0)$,

$$[x] \cdot [y] > \varphi(0)$$

2) If $[x] < \varphi(0)$, $[y] > \varphi(0)$,

$$[x] \cdot [y] < \varphi(0).$$

We now know that $\mathbb{X}$, with "$+$", "$\cdot$", and "$\leq$", satisfies all algebraic and order properties of $\mathbb{R}$ and contains a copy of $\mathbb{Q}$.

It remains to prove the

Completeness Axiom of $\mathbb{R}$:

$\forall\ S \subseteq \mathbb{X}$, S bounded above, $\sup(S)$ exists in $\mathbb{X}$.

Note that we aim to prove the completeness axiom, so that it is actually no longer an axiom! First, a preliminary.

Lemma: Let $S \subseteq \mathbb{X}$, S bounded above. Then $\exists\ p, q \in \mathbb{Q}$, $\varphi(p)$ is an upper bound for S and $\exists\ [z] \in S$, $\varphi(q) < [z]$.

proof: Since S is bounded above, $\exists\ [x] \in \mathbb{X}$, $x = (x_n)_{n=1}^{\infty}$, with

$$[x] > [y] \quad \forall\ [y] \in S.$$

Since $(x_n)_{n=1}^{\infty}$ is Cauchy,

$(x_n)_{n=1}^{\infty}$ is bounded, so

$\exists\, M \in \mathbb{Q},\ M > 0,$

$$|x_n| < M \quad \forall\, n \in \mathbb{N}.$$

Let $p = M+1$. Then

$$p - x_n = M + 1 - x_n > 1$$

$\forall\, n \in \mathbb{N} \Rightarrow \varphi(p) > [x],$

and so $\varphi(p) > [y]$ $\forall$

$[y] \in \underline{X}$ ✓

Now choose $[z] \in S$,

$z = (z_n)_{n=1}^{\infty}$. Since $(z_n)_{n=1}^{\infty}$

is Cauchy, $\exists$ $K \in \mathbb{Q}$, $K > 0$,

$|z_n| < K$ $\forall$ $n \in \mathbb{N}$.

Let $q = -K - 1$. Then

$$z_n - q = z_n - (-K-1) > 1$$

$$\Rightarrow \varphi(q) < [z]. \quad \square$$

Let S, p, q be as in the statement of the previous lemma.

Set $x_1 = p$ and $y_1 = q$.

Let $$z_1 = \frac{x_1 + y_1}{2}$$

If $\varphi(z_1)$ is an upper bound for S, set

$$x_2 = z_1 \quad \text{and}$$

$$y_2 = y_1.$$

If $\varphi(z_1)$ is not an upper bound for S, set

$$x_2 = x_1$$
$$y_2 = z_1.$$

Continuing in this manner, for $n \in \mathbb{N}$, let

$$z_n = \frac{x_n + y_n}{2} \text{ and}$$

if x_n, y_n already chosen, let

$$x_{n+1} = \begin{cases} x_n, & \varphi(z_n) \text{ not an upper bound for } S \\ z_n, & \varphi(z_n) \text{ an upper bound for } S \end{cases}$$

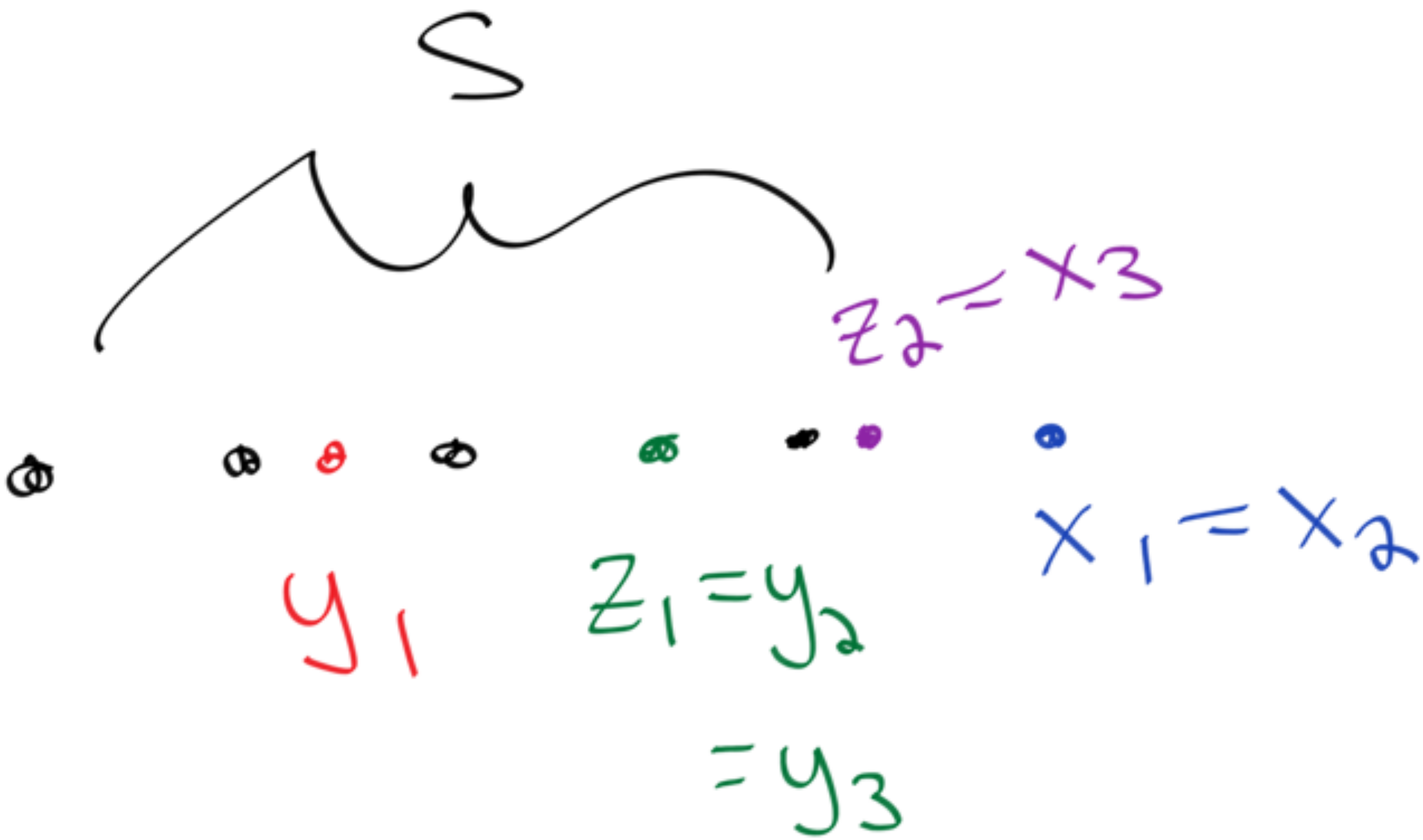
$$y_{n+1} = \begin{cases} y_n, & \varphi(z_n) \text{ an upper bound for } S \\ z_n, & \varphi(z_n) \text{ not an upper bound for } S \end{cases}$$

We obtain two sequences of rationals, $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that

i) $\varphi(x_n)$ is an upper bound for S $\forall n \in \mathbb{N}$

ii) $\varphi(y_n)$ is not an upper bound for S $\forall n \in \mathbb{N}$

Picture



$y_n, x_n \to \sup(S)$

We need to show:

1) $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ are Cauchy sequences of rationals

2) $[(x_n)_{n=1}^{\infty}]$ is an upper bound for S

3) $[(x_n)_{n=1}^{\infty}] = [(y_n)_{n=1}^{\infty}]$, which will imply $[(x_n)_{n=1}^{\infty}] = \sup(S)$.

Proof: 1) We know

$x_1 > y_1$, so

$x_1 - y_1 > 0$.

If $x_2 = z_1$, then

$y_2 = y_1$, and

$$x_2 - y_2 = \frac{x_1 + y_1}{2} - y_1$$

$$= \frac{x_1 - y_1}{2} > 0.$$

If $y_2 = z_1$, then

$x_2 = x_1$, and

$$x_2 - y_2 = x_1 - \left(\frac{x_1 + y_1}{2}\right)$$

$$= \frac{x_1 - y_1}{2}$$

Inductively assume

$$x_n - y_n = \frac{x_1 - y_1}{2^{n-1}}.$$

Show $x_{n+1} - y_{n+1} = \dfrac{x_1 - y_1}{2^n}$

If $z_n = x_{n+1}$, then

$y_{n+1} = y_n$ and

$$x_{n+1} - y_{n+1} = \frac{x_n + y_n}{2} - y_n$$

$$= \frac{x_n - y_n}{2}$$

induction $\rightarrow$ $$= \frac{\left(\frac{x_1 - y_1}{2^{n-1}}\right)}{2}$$

$$= \frac{x_1 - y_1}{2^n} \checkmark$$

If $z_n = y_{n+1}$, then

$x_{n+1} = x_n$, so

$$x_{n+1} - y_{n+1} = x_n - \left(\frac{x_n + y_n}{2}\right)$$

$$= \frac{x_n - y_n}{2}$$

$$= \frac{\left(\frac{x_1 - y_1}{2^{n-1}}\right)}{2}$$

$$= \frac{x_1 - y_1}{2^n} \quad \checkmark$$

We have proved our claim.

Observe now that if

$x_{n+1} = z_n$, then

$$x_{n+1} - x_n = \frac{x_n + y_n}{2} - x_n$$

$$= \frac{y_n - x_n}{2}$$

$$= \frac{y_1 - x_1}{2^n}$$

If $y_{n+1} = z_n$, then

$x_{n+1} = x_n$ and

$x_{n+1} - x_n = 0$, so

$$|x_{n+1} - x_n| \leq \frac{x_1 - y_1}{2^n}$$

$\forall\, n \in \mathbb{N}$

WLOG, assume $m > n$.

Then $|x_m - x_n|$

$$= \left| \sum_{i=1}^{m-n} \left(x_{n+i} - x_{n+i-1} \right) \right|$$

$$\leq \sum_{i=1}^{m-n} \frac{|y_1 - x_1|}{2^{n+i-1}}$$

$$= |y_1 - x_1| \sum_{i=1}^{m-n} \frac{1}{2^{n+i-1}}$$

$$= |y_1 - x_1| \sum_{i=n}^{m-1} \frac{1}{2^i}$$

Now $\displaystyle\sum_{i=n}^{m-1} \frac{1}{2^i} < \frac{1}{2^{n-1}}$

(proof after break!), so

$$|x_m - x_n| = x_1 - y_1 \sum_{i=n}^{m-1} \frac{1}{2^i}$$

$$< \frac{x_1 - y_1}{2^{n-1}}$$

By the archimedean property of $\mathbb{N}$, if $p \in \mathbb{Q}, p > 0, \exists$ $N \in \mathbb{N}, \frac{x_1 - y_1}{2^{N-1}} < p$.

Hence, $\forall\, m, n \geq N$,

$$|x_m - x_n| < \frac{x_1 - y_1}{2^{N-1}}$$

$$< p$$

$\Rightarrow (x_n)_{n=1}^{\infty}$ is Cauchy.

Similarly,

$$|y_{n+1} - y_n| \leq \frac{x_1 - y_1}{2^{n-1}}$$

$\Rightarrow$ $(y_n)_{n=1}^{\infty}$ is Cauchy.

This proves 1).

3) We know

$$x_n - y_n = \frac{x_1 - y_1}{2^{n-1}}$$

$$\Rightarrow \lim_{n \to \infty} |x_n - y_n|$$

$$= \lim_{n \to \infty} \frac{|x_1 - y_1|}{2^{n-1}}$$

$$= 0.$$

Therefore $(x_n)_{n=1}^{\infty}$

$\sim (y_n)_{n=1}^{\infty}$

2) Now to show

$$\left[(x_n)_{n=1}^{\infty}\right] \text{ is}$$

an upper bound for

S. Suppose not.

Then $\exists\ [w] \in S$,

$$[w] > \left[(x_n)_{n=1}^{\infty}\right].$$

This means:

if $w = (w_n)_{n=1}^{\infty}$

$\exists \ r > 0, \ r \in \mathbb{Q}$

and $N_1 \in \mathbb{N}$,

$$\boxed{w_n - x_n > r}$$

$\forall \ n \geq N_1$.

But $\varphi(x_n)$ is an upper bound for S $\forall \ n \in \mathbb{N} \Rightarrow$

$\forall\ n \in \mathbb{N}$, either

$\varphi(x_n) \sim [w]$

or $\exists\ r_n > 0,\ r_n \in \mathbb{Q}$

$K_n \in \mathbb{N}$,

$$x_n - w_i > r_n$$

$\forall\ i \geq K_n$.

Now if $[\omega] = \varphi(x_n)$

for some $n \in \mathbb{N}$, then

since $\varphi(x_k)$ is an

upper bound for S

and $x_{k+1} \leq x_k \; \forall \; k \in \mathbb{N}$,

then $x_k = x_n \; \forall \; k \geq n$

$$\Rightarrow \; [(x_n)_{n=1}^{\infty}] = \varphi(x_k)$$

$$= [\omega]$$

Finally, suppose $[w] < \varphi(x_n)$ $\forall\ n \in \mathbb{N}$. Since $(w_n)_{n=1}^{\infty}$ is Cauchy, $\exists\ N_2 \in \mathbb{N}$,

$$\boxed{|w_n - w_m| < \frac{r}{2}} \ \forall\ n, m \geq N_2.$$

Let $N = \max\{N_1, N_2\}$

Then $\forall\ m, n \geq N$

$$\begin{aligned} w_m - x_n &= w_m - w_n + w_n - x_n \\ &> -\frac{r}{2} + r = \frac{r}{2} \end{aligned}$$

However, fixing n,

$x_n - w_i > r_n > 0$

$\forall\ i \geq K_n$

$\Rightarrow \forall\ m \geq \max\{K_n, N\}$,

$$\boxed{\frac{r}{2} < w_m - x_n < -r_n},$$

Contradiction.

Therefore, $[w] \leq [(x_n)_{n=1}^{\infty}]$

$\Rightarrow [(x_n)_{n=1}^{\infty}]$ is an upper bound for S.

Now if $[t] \in \mathbb{X}$, $[t]$ is an upper bound for S, then since $\varphi(y_n)$ is not an upper bound for S, $\varphi(y_n) < [t]$ $\forall n \in \mathbb{N}$.

We suppose $[t] < \left[(y_n)_{n=1}^{\infty}\right]$.

$$= \left[(x_n)_{n=1}^{\infty}\right].$$

Then $\exists M_1 \in \mathbb{N}$, $b \in \mathbb{Q}$,

$$\boxed{y_n - t_n > b > 0}$$

$\forall n \geq M_1$.

Now since $(t_n)_{n=1}^{\infty}$ is Cauchy, $\exists\ M_2 \in \mathbb{N}$,

$$\boxed{|t_n - t_m| < \frac{b}{2}} \ \forall$$

$n, m \geq M_2$.

Hence, $\forall\ n, m \geq M = \max\{M_1, M_2\}$,

$$\begin{aligned} y_n - t_m &= y_n - t_n + t_n - t_m \\ &> b - \frac{b}{2} \\ &= \frac{b}{2} \end{aligned}$$

Now as before, since

$\varphi(y_n)$ is not an upper

bound for S, $\forall\ n \in \mathbb{N}$,

$\exists\ L_n \in \mathbb{N}$ and $v_n \in \mathbb{Q}$,

$$\boxed{t_i - y_n > v_n > 0}$$

$\forall\ i \geq L_n$.

Then with n fixed and

$m > \max\{L_n, M\}$,

$t_m - y_n > u_n > 0$ and

$y_n - t_m > \frac{b}{2} > 0$

$\Rightarrow u_n < t_m - y_n < -\frac{b}{2}$

contradiction. Therefore

$[t] \geq [(y_n)_{n=1}^{\infty}] = [(x_n)_{n=1}^{\infty}]$

$\Rightarrow [(x_n)_{n=1}^{\infty}] = \sup(S)$