Constructing $\mathbb{N}$

Starting Point: Sets and elements

We will not define either sets or elements, but the ZFC axioms will tell us what sets and elements act like.

Zermelo-Fraenkel-Choice (ZFC) Axioms

These are the axioms that govern set and element behavior in this class. A full list of these axioms is on Canvas, but I will write down the ones that we need.

Naive assumption: $\exists$ a set!

The Axioms We Will Need

1) Axiom of Extensionality (ZFC 1)

Two sets S and T are equal if and only if they have the same elements.

2) Axiom Schema of Separation (ZFC 3)

Let S be a set. Let φ be a logical formula (i.e. a sentence) involving a free variable x.

$$\boxed{\text{Then } \{x \mid \varphi(x) \in S\} \text{ is a set.}}$$

Usually, we will choose x from a set T.

This is called a "schema" since there is one axiom for each φ!

Definition: (empty set) Let S be a set. The **empty set** of S is

$$\boxed{\phi = \{x \in S \mid x \neq x\}}.$$

By the Axiom Schema of Separation, $\phi \subseteq S$.

Now if T is any other set, the Axiom of Extensionality tells us that the empty set of T equals the empty set of S.

Consequently, we are justified in saying that $\emptyset \subseteq T \ \forall$ sets T, and in using the terminology *the* empty set.

3) Axiom of Union (if you need it to be happy)

Let S and T be sets.

Then

$$\boxed{S \cup T = \{ x \mid (x \in S) \vee (x \in T) \}}$$

is a set.

The Elements of $\mathbb{N}$

Contrary to what will usually be the case in this class, we will follow the convention that $0 \in \mathbb{N}$.

In fact, we will set

- $0 = \emptyset$
- $1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$

  (i.e. a set with 1 element...)
- $2 = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\}$

  (i.e. a set with 2 elements)

- $3 = 2 \cup \{2\}$

$$= \left(\{\emptyset\} \cup \{\{\emptyset\}\}\right) \cup \{\{\emptyset\} \cup \{\{\emptyset\}\}\}$$

(i.e. a set with 3 elements)

In this manner,

- $4 = 3 \cup \{3\}$
- $5 = 4 \cup \{4\}$
- $6 = 5 \cup \{5\}$

etc.

But what does "etc." mean?
When we do "etc.", do we get a set?

Notation: (successor) In the language we just constructed, the successor of a set S is defined as $S \cup \{S\}$, and denoted by $\boxed{S^+ = S \cup \{S\}}$

We want to define $\mathbb{N}$ as the set containing 0 $(= \emptyset)$ and all successors of its elements.
Why is this a set?

4) Axiom of Infinity

$\exists$ an infinite set S such that $\emptyset \in S$ and such that if $x \in S$, then $x^+ \in S$.

Note: any set S with the property that $x^+ \in S$ whenever $x \in S$ will be called a Sucessor set.

Definition: (N) Let S be any successor set and $\phi \in S$.

We define

$$\boxed{\mathbb{N} = \bigcap_{\substack{S \text{ a successor set},\ \phi \in S}} S}$$

We intersect all successor sets.

If you like, you may include another axiom specifying that intersections of sets are sets.

In order to be satisfied with this definition, we'd need to check that

1) the intersection of socessor sets is a socessor set. (HW 1).

2) $\mathbb{N}$ is the smallest socessor set with $\emptyset \in \mathbb{N}$. (almost immediate from definition)

This seems like a rather artificial construction. We want to make sure that $\mathbb{N}$, as we've defined it, has the desired properties we're used to. Those properties come from the Peano Axioms.

The Peano Axioms

Axioms that determine $\mathbb{N}$

1) $0 \in \mathbb{N}$

2) $\forall\, n \in \mathbb{N}$, $n^+ \in \mathbb{N}$

3) (Principle of Mathematical Induction)

If $S \subseteq \mathbb{N}$, $0 \in S$, and $n^+ \in S$ if $n \in S$, then $S = \mathbb{N}$.

4) If $n \in \mathbb{N}$, $n^+ \neq 0$.

5) If $n, m \in \mathbb{N}$ and $n^+ = m^+$,

then $n = m$.

We must prove 4) and 5)!