

The Peano Axioms

Axioms that determine \mathbb{N}

1) $0 \in \mathbb{N}$ ✓

2) $\forall n \in \mathbb{N}, n^+ \in \mathbb{N}$ ✓

3) (Principle of Mathematical Induction)

If $S \subseteq \mathbb{N}$, $0 \in S$, and $n^+ \in S$

if $n \in S$, then $S = \mathbb{N}$. ✓

4) If $n \in \mathbb{N}$, $n^+ \neq 0$.

5) If $n, m \in \mathbb{N}$ and $n^+ = m^+$,
then $n = m$.

We must prove 4) and 5)!

Proof of 4) By contradiction.

Suppose $\exists n \in \mathbb{N}$ with

$$n^+ = O = \emptyset.$$

But $n^+ = n \cup \{n\}$.

So

$$n \cup \{n\} = n^+ = \emptyset$$

[But then $n \in \emptyset$,
contradiction.] Therefore,

there is no $n \in \mathbb{N}$ with

$$n^+ = O.$$

To prove 5), we will
first require two lemmas!

Lemma = something technical
you need to prove
a result (usually
the result is a
theorem of some
kind).

Lemma: Let $n \in \mathbb{N}$ and $x \in n$.

Then n is not a subset
of x .

proof: We want to use the
Principle of Mathematical

Induction :

$$S = \{n \in \mathbb{N} \mid \text{if } x \in n, n \notin x\}.$$

Want to show

$$1) 0 \in S$$

$$2) n+ \in S \text{ whenever } n \in S$$

The principle will then tell us
that $S = \mathbb{N}$

1) $0 \in S$ Since $0 = \emptyset$ and there are no elements $x \in \emptyset$, the condition holds vacuously, so $0 \in S$!

2) If $n \in S$, then $n^+ \in S$

Want to show: if $x \in n^+$,

then $n^+ \notin x$ assuming

that if $x \in n$, then $n \notin x$.

Suppose $x \in n^+ = n \cup \{n\}$.

Then either

(i) $x \in n$. By induction,

$n \notin X$. But $n \subseteq n^+$,

so $n^+ \notin X$.

or

(ii) $x \in \{n\}$. Then $x = n$.

Since $n^+ = n \cup \{n\}$,

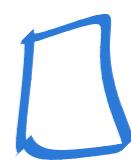
$n^+ \notin X$.

We've shown that $0 \in S$ and whenever $n \in S$, then $n+1 \in S$. By the Principle of Mathematical Induction,

$S = \mathbb{N}$. Then

$$\mathbb{N} = S = \{n \in \mathbb{N} \mid \text{if } x \in n, n \notin x\},$$

so **every** $n \in \mathbb{N}$ has the property that if $x \in n$, $n \notin x$.



Lemma: Let $n \in \mathbb{N}$ and $x \in n$. Then

$$x \subseteq n.$$

Proof: Again by induction.

Let

$$S = \{n \in \mathbb{N} \mid \text{if } x \in n, \text{ then } x \subseteq n\}.$$

We want to show

1) $0 \in S$

2) If $n \in S$, then $n^+ \in S$.

1) $0 \in S$

Remember $0 = \emptyset$

and so there do not exist

$x \in \emptyset$, so the condition

defining S holds vacuously!

2) If $n \in S$, then $n^+ \in S$.

We want to show that

if $x \in n^+$, then $x \subseteq n^+$

assuming that if $x \in n$,

then $x \subseteq n$.

Let $x \in n^+$. Then either

i) $x \in n$, in which case,

our inductive assumption

gives $x \subseteq n \subseteq n^+ = n \cup \{n\}$.

Therefore, $x \subseteq n^+$.

or

ii) $x \in \{n\}$. But then

$x = n \subseteq n^+ = n \cup \{n\}$.

We have then shown that if

$n \in S$, then $n^+ \in S$.

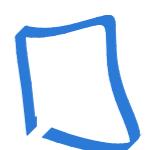
So by the Principle of Mathematical Induction,
 $S = \mathbb{N}$. But recall that

$$\mathbb{N} = S = \{n \in \mathbb{N} \mid \text{if } x \in n, \text{ then } x \in n\},$$

so we've established that

every $n \in \mathbb{N}$ has the property

that if $x \in n$, then $x \in n$.



Proof of 5) We start off with
 $n, m \in \mathbb{N}$. We want
to show that if

$$n^+ = m^+, \text{ then}$$

$$n = m.$$

Since $n^+ = m^+$,

$$n \cup \{n\} = m \cup \{m\}.$$

Want to show $m = n$ by

Showing $m \subseteq n$ and

$$n \subseteq m.$$

Since $n \subseteq n^+ = m^+$, we know

$$n \subseteq m \cup \{m\}.$$

So either

1) $m \in n$

By our lemmas,

$$m \subseteq n \text{ (second lemma)}$$

and $n \not\subseteq m$.

But we know $m \subseteq n$ from
assuming $m \in n$. Therefore,

$$m \cup \{m\} \subseteq n$$

$$= m^+ = n^+ \text{ by assumption.}$$

This yields

$$\Lambda^+ \subseteq \Lambda$$

But $\Lambda^+ \subseteq \Lambda$ contradicts our

first lemma since

$$\Lambda^+ = \Lambda \cup \{\infty\} \quad \text{and} \quad \infty \in \Lambda^+.$$

The lemma says that if

$$\infty \in \Lambda^+, \text{ then } \infty \notin \Lambda$$

So we have a contradiction.

We arrived here by initially

assuming $\infty \in \Lambda$, so this

assumption must be false!

Hence, we know that

2) $m \notin n$

But then

$$n \subseteq n^+ = n^+ = n \cup \{m\}$$

Since $m \notin n$, $\{m\} \notin n$, so

$$n \subseteq m \quad \checkmark$$

By interchanging the roles of m and n , we can show $m \subseteq n$.

We then conclude $m = n$.



Conclusion: Our strange set-theoretic definition of \mathbb{N} satisfies all the Peano Axioms! We would like to define the standard operations of addition, multiplication, etc. on \mathbb{N} using induction. We need a recursion result in order to do so!

Theorem: (Recursion) Let S be a set and let $x \in S$, $f: S \rightarrow S$. Then $\exists!$ $g: \mathbb{N} \rightarrow S$ such that $g(0) = x$ and $g(n^+) = f(g(n))$ $\forall n \in \mathbb{N}$.

Note: " $\exists!$ " means "There exists a unique"

How we use this result to get,

say, addition: $S = \mathbb{N}$

$$f(n) = n^+ \quad \forall n \in \mathbb{N}.$$

If $m \in \mathbb{N}$. By Theorem, $\exists!$

$g_m: \mathbb{N} \rightarrow \mathbb{N}$ w/ $g_m(0) = m$ and

$$g_m(n^+) = f(g_m(n)) = (g_m(n))^+$$

So for example if $n=6$,

$$g_6(0) = 6$$

$$g_6(1) = g_6(0^+)$$

$$= (g_6(0))^+$$

$$= 6^+$$

$$= 7$$

$$g_6(2) = g_6(1^+)$$

$$= (g_6(1))^+$$

$$= 7^+$$

$$= 8$$

We define $6+n = g_6(n)$

proof: later...

Definition: (addition) Let $m, n \in \mathbb{N}$.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = n^+$.

We induce, for each $m \in \mathbb{N}$,

a function $g_m: \mathbb{N} \rightarrow \mathbb{N}$

from the Recursion Theorem

such that

$$1) \ g_m(0) = m$$

$$\begin{aligned} 2) \ g_m(n^+) &= f(g_m(n)) \\ &= (g_m(n))^+ \end{aligned}$$

Then we define

$$\boxed{m + n = g_m(n)} \quad \forall n \in \mathbb{N}$$

- Theorem:
- 1) Addition on \mathbb{N} is associative.
 - 2) Addition on \mathbb{N} is commutative.

Proof: Start 1): We want to show that , $\forall m, n, k \in \mathbb{N}$,

$$(m+n)+k = m+(n+k)$$

Use induction on k .

Let

$$S = \{k \in \mathbb{N} \mid (m+n)+k = m+(n+k)\}.$$

Step 1 : OES If $k=0$, the

Statement is

$$(m+n)+0 = m+(n+0)$$

$$(m+n)+0 = g_{m+n}(0) = m+n$$

Similarly, $n+0 = g_n(0) = n$,

So we get

$$(m+n)+0 = m+n = m+(n+0) \checkmark$$

So OES !