# The Peano Axioms

Axioms that determine $\mathbb{N}$

1) $0 \in \mathbb{N}$ ✓

2) $\forall\ n \in \mathbb{N},\ n^+ \in \mathbb{N}$ ✓

3) (Principle of Mathematical Induction)

If $S \subseteq \mathbb{N}$, $0 \in S$, and $n^+ \in S$ if $n \in S$, then $S = \mathbb{N}$. ✓

4) If $n \in \mathbb{N}$, $n^+ \neq 0$.

5) If $n, m \in \mathbb{N}$ and $n^+ = m^+$, then $n = m$.

We must prove 4) and 5)!

Proof of 4) By contradiction.

Suppose $\exists\ n \in \mathbb{N}$ with

$n^+ = 0 = \emptyset$.

But $n^+ = n \cup \{n\}$.

So

$n \cup \{n\} = n^+ = \emptyset$

[But then $n \in \emptyset$, contradiction.] Therefore, there is no $n \in \mathbb{N}$ with

$n^+ = 0$.

To prove 5), we will first require two lemmas!

Lemma = something technical you need to prove a result (usually the result is a theorem of some kind).

**Lemma:** Let $n \in \mathbb{N}$ and $x \in n$.

Then $n$ is not a subset of $x$.

**proof:** We want to use the Principle of Mathematical Induction:

$$S = \{n \in \mathbb{N} \mid \text{if } x \in n,\ n \notin x\}.$$

Want to show

1) $0 \in S$

2) $n^+ \in S$ whenever $n \in S$

The principle will then tell us that $S = \mathbb{N}$

1) $0 \in S$ Since $0 = \emptyset$ and

there are no elements $x \in \emptyset$,

the condition holds vacuously,

So $0 \in S$!

2) If $n \in S$, then $n^+ \in S$

Want to show: if $x \in n^+$,

then $n^+ \notin x$ assuming

that if $x \in n$, then $n \notin x$.

Suppose $x \in n^+ = n \cup \{n\}$.

Then either

(i) $x \in n$. By induction, $n \notin x$. But $n \subseteq n^+$, so $n^+ \notin x$.

or

(ii) $x \in \{n\}$. Then $x = n$. Since $n^+ = n \cup \{n\}$, $n^+ \notin x$.

We've shown that $0 \in S$ and whenever $n \in S$, then $n^+ \in S$. By the Principle of Mathematical Induction, $S = \mathbb{N}$. Then

$$\mathbb{N} = S = \{n \in \mathbb{N} \mid \text{if } x \in n, n \notin x\},$$

so every $n \in \mathbb{N}$ has the property that if $x \in n$, $n \notin x$.

$\square$

**Lemma:** Let $n \in \mathbb{N}$ and $x \in n$. Then

$x \subseteq n$.

**Proof:** Again by induction.

Let

$$S = \{ n \in \mathbb{N} \mid \text{if } x \in n, \text{ then } x \subseteq n \}.$$

We want to show

1) $0 \in S$

2) If $n \in S$, then $n^+ \in S$.

1) $0 \in S$ Remember $0 = \emptyset$

and so there do not exist

$x \in \emptyset$, so the condition

defining $S$ holds vacuously!

2) If $n \in S$, then $n^+ \in S$.

We want to show that

if $x \in n^+$, then $x \subseteq n^+$

assuming that if $x \in n$,

then $x \subseteq n$.

Let $x \in n^+$. Then either

i) $x \in n$, in which case,

our inductive assumption

gives $x \subseteq n \subseteq n^+ = n \cup \{n\}$.

Therefore, $x \subseteq n^+$.

or

ii) $x \in \{n\}$. But then

$$x = n \subseteq n^+ = n \cup \{n\}.$$

We have then shown that if

$n \in S$, then $n^+ \in S$.

So by the Principle of Mathematical Induction, $S = \mathbb{N}$. But recall that

$$\mathbb{N} = S = \{n \in \mathbb{N} \mid \text{if } x \in n \text{, then } x \subseteq n\},$$

So we've established that every $n \in \mathbb{N}$ has the property that if $x \in n$, then $x \subseteq n$.

$\square$

Proof of 5) We start off with $n, m \in \mathbb{N}$. We want to show that if $n^+ = m^+$, then $n = m$.

Since $n^+ = m^+$,

$$n \cup \{n\} = m \cup \{m\}.$$

Want to show $m = n$ by showing $m \subseteq n$ and $n \subseteq m$.

Since $n \subseteq n^+ = m^+$, we know

$n \subseteq m \cup \{m\}$.

So either

1) $m \in n$ By our lemmas,

$m \subseteq n$ (second lemma)

and $n \notin m$.

But we know $m \subseteq n$ from assuming $m \in n$. Therefore,

$$m \cup \{m\} \subseteq n$$

$$= m^+ = n^+ \text{ by assumption.}$$

This yields

$$n^+ \subseteq n.$$

But $n^+ \subseteq n$ contradicts our first lemma since

$$n^+ = n \cup \{n\} \quad \text{and} \quad n \in n^+.$$

The lemma says that if $n \in n^+$, then $n^+ \not\subseteq n$

So we have a contradiction.

We arrived here by initially assuming $n \in n$, so this assumption must be false!

Hence, we know that

2) $m \notin n$  But then

$$n \subseteq n^+ = m^+ = m \cup \{m\}$$

Since $m \notin n$, $\{m\} \notin n$, so

$n \subseteq m$ ✓

By interchanging the roles of $m$ and $n$, we can show $m \subseteq n$.

We then conclude $m = n$.

$\square$

**Conclusion:** Our strange set-theoretic definition of $\mathbb{N}$ satisfies all the Peano Axioms! We would like to define the standard operations of addition, multiplication, etc. on $\mathbb{N}$ using induction. We need a recursion result in order to do so!

**Theorem:** (Recursion) Let $S$ be a set and let $x \in S$, $f: S \to S$. Then $\exists!\ g: \mathbb{N} \to S$ such that $g(0) = x$ and $g(n^+) = f(g(n))$ $\forall\ n \in \mathbb{N}$.

**Note:** "$\exists!$" means "There exists a unique"

How we use this result to get, say, addition: $S = \mathbb{N}$

$f(n) = n^+$ $\forall n \in \mathbb{N}$.

If $m \in \mathbb{N}$. By Theorem, $\exists!$ $g_m: \mathbb{N} \to \mathbb{N}$ w/ $g_m(0) = m$ and $g_m(n^+) = f(g_m(n)) = (g_m(n))^+$

So for examples, if $m=6$,

$$g_6(0) = 6$$

$$\begin{aligned} g_6(1) &= g_6(0^+) \\ &= (g_6(0))^+ \\ &= 6^+ \\ &= 7 \end{aligned}$$

$$\begin{aligned} g_6(2) &= g_6(1^+) \\ &= (g_6(1))^+ \\ &= 7^+ \\ &= 8 \end{aligned}$$

We define $6+n = g_6(n)$

proof: later...

**Definition:** (addition) Let $m, n \in \mathbb{N}$.

Let $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n^+$.

We induce, for each $m \in \mathbb{N}$, a function $g_m: \mathbb{N} \to \mathbb{N}$ from the Recursion Theorem such that

1) $g_m(0) = m$

2) $g_m(n^+) = f(g_m(n)) = (g_m(n))^+$.

Then we define

$$\boxed{m + n = g_m(n)} \quad \forall\, n \in \mathbb{N}$$

**Theorem:** 1) Addition on $\mathbb{N}$ is associative.

2) Addition on $\mathbb{N}$ is commutative.

**proof:** Start 1) : We want to

Show that, $\forall$ $m, n, k \in \mathbb{N}$,

$$(m+n)+k = m+(n+k)$$

Use induction on $k$.

Let

$$S = \{k \in \mathbb{N} \mid (m+n)+k = m+(n+k)\}.$$

Step 1: OES   If $k=0$, the

statement is

$$(m+n)+0 = m+(n+0)$$

$$(m+n)+0 = g_{m+n}(0) = m+n$$

Similarly, $n+0 = g_n(0) = n$,

so we get

$$(m+n)+0 = m+n = m+(n+0)$$

So OES!