Math 452/552 Problem Set 1

To Be Presented Friday, January 18

1) (5 points) Let (X,d) be a metric space and let $f:(X,d)\to(\mathbb{R},|\cdot|)$ be continuous. Let

$$\mathcal{Z}(f)=\{x\in X \mid f(x)=0\}.$$

Prove that $\mathcal{Z}(f)$ is closed.

2) (5 points) Prove that the monotone convergence theorem implies the nested interval property.

3) (10 points) Let $x_1=\dfrac{1}{4}$ and for $n>1$, define $x_n=\left(\dfrac{1}{4}\right)^{x_{n-1}}$. Prove that $(x_n)_{n=1}^{\infty}$ converges.

4) (5 points) Let $f:\mathbb{R}\to\mathbb{R}$ and suppose for all $x,y\in\mathbb{R}$,

$$|f(x)-f(y)|<(x-y)^2.$$

Show that there exists $c\in\mathbb{R}$ such that $f(x)=c$ for all $x\in\mathbb{R}$, i.e., f is constant.

5) (5 points) Show that if $a_n>0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}(n^2a_n)$ exists, then $\sum_{n=1}^{\infty}a_n$ converges.

6) (15 points) Let $f:[a,b]\to\mathbb{R}$ be continuous in the usual metric on both its domain and range. Suppose that for any point $x\in[a,b]$, there is another point $x'\in[a,b]$ such that $|f(x')|\le|f(x)|/2$. Prove that there exists a point $x_0\in[a,b]$ where $f(x_0)=0$.

7) (15 points) Let (X,d) be a compact metric space and let $f:X\to Y\subset X$ satisfy

$$d(f(x),f(y))=d(x,y)$$

for all $x,y\in X$. Prove that $Y=X$.

8) (10 points) Explain Conway's base-13 counterexample to the converse of the intermediate value theorem; show that it is nowhere-continuous and satisfies the conclusion to the intermediate value theorem.

9) (5 points) Prove that if $f : [a, b] \to \mathbb{R}$ satisfies the conclusion of the intermediate value theorem and is increasing, then f must be continuous.

10) (5 points) Show that $f(x) = 1/x^2$ is uniformly continuous on $(1, \infty)$.

11) (5 points) Show that if $f : [a.b] \to \mathbb{R}$ is bounded, then $U(f) \geq L(f)$.

12) (Exercise 7.2.6- 5 points) Suppose $f : [a, b] \to \mathbb{R}$ is increasing. Prove that f is Riemann-integrable.

13) (5 points) Exercise 7.3.4.

14) (10 points) Exercise 7.3.6.