Math 452/552 Problem Set 2

To Be Presented Friday, February 1

1) (5 points) (Exercise 7.6.4) Show that the Cantor set has Lebesgue measure zero.

2) (5 points) If C denotes the Cantor set. For any subset $S \subseteq \mathbb{R}$, we define

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

Show that χ_C is Riemann integrable on $[0, 1]$.

3) (5 points) (Exercise 7.4.1) If f is Riemann integrable on $[a, b]$, show that $|f|$ is Riemann integrable on $[a, b]$ and that

$$\int_a^b |f| \, dx \geq \left| \int_a^b f \, dx \right|.$$

4) (5 points each part) Exercise 7.5.8.

5) (5 points) Suppose $f \geq 0$ and continuous on $[a, b]$. If $\displaystyle\int_a^b f \, dx = 0$, show that $f = 0$ identically on $[a, b]$.

6) Let p and q be positive real numbers with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

a) (10 points) Prove that if $a, b \geq 0$ are real numbers, then $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$, with equality iff $a^p = b^q$.

b) (5 points) If $f, g \geq 0$ on $[a, b]$ and

$$\int_a^b f^p \, dx = \int_a^b g^q \, dx = 1,$$

show $\displaystyle\int_a^b fg \, dx \leq 1$.

c) (10 points) Prove Hölder's Inequality:

$$\left| \int_a^b fg \ dx \right| \leq \left\{ \int_a^b |f|^p \ dx \right\}^{1/p} \left\{ \int_a^b |g|^q \ dx \right\}^{1/q}.$$

7) For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

When s is complex, this is known as the *Riemann zeta function*. We will only consider reals, though. For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to x.

a) (10 points) Prove that $\zeta(s) = s \displaystyle\int_1^{\infty} \frac{[x]}{x^{s+1}} \ dx.$

b) (5 points) Prove that $\zeta(s) = \dfrac{s}{s-1} - s \displaystyle\int_1^{\infty} \frac{x - [x]}{x^{s+1}} \ dx.$

8) (15 points) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x) \ dx = \int_0^1 f(x)(x^n + x^{n+2}) \ dx$$

for all $n \in \mathbb{N} \cup \{0\}$. Show that $f = 0$ identically on $[0,1]$.

9) (15 points) Let

$$X = \{f : [0, 2\pi] \to \mathbb{R} \mid f \text{ is continuous and } |f| \leq 1 \ \forall x \in [0, 2\pi]\}.$$

Define a metric d on X by

$$d(f,g) = \sqrt{\int_0^{2\pi} (f-g)^2 \ dx}.$$

Is (X, d) a compact metric space? Prove your assertion.