Math 452/552 Problem Set 3

To Be Presented Wednesday, February 20

1) (5 points) (Exercise 7.5.5) Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a,b]$. Assuming each f_n' is continuous, we can apply Theorem 7.5.1 (i) to get

$$\int_a^x f_n' = f_n(x) - f_n(a)$$

for all $x \in [a,b]$. Show that $g(x) = f'(x)$.

2) (5 points) (Exercise 6.4.4) In Section 5.4, we postponed the argument that the nowhere-differentiable function

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

is continuous on $\mathbb{R}$. Use the Weierstrass M-Test to supply the missing proof.

3) (5 points) (Exercise 6.3.1) a) If

$$h_n(x) = \frac{\sin(nx)}{n},$$

show that $h_n \to 0$ uniformly on $\mathbb{R}$. At what points does the sequence of derivatives converge?

4) (10 points, all or nothing) Exercise 6.2.12

5) (5 points, each part) Exercise 6.2.11

6) (5 points) (Exercise 6.2.3) Consider the sequence of function

$$h_n(x) = \frac{x}{1+x^n}.$$

on $[0,\infty)$. Determine the pointwise limit of $(h_n)_{n=1}^{\infty}$ and show that the convergence cannot be uniform. Find a subset of $[0,\infty)$ that is not measure-zero and on which the convergence is uniform.

7) (10 points) Suppose g and $(f_n)_{n=1}^{\infty}$ are real-valued functions defined on $(0,\infty)$ which are Riemann-integrable on $[t,T]$ whenever $0 < t < T < \infty$,

$|f_n| \le g$ for all $x \in (0, \infty)$, and $f_n \to f$ uniformly on every compact subset of $(0, \infty)$. Suppose

$$\int_0^\infty g(x)\, dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx.$$

This is a weak form of Lebesgue's Dominated Convergence Theorem which, by the way, you cannot quote to solve this problem!

8) (10 points) Let $(f_n)_{n=1}^\infty$ be a sequence of real-valued continuous functions which converge uniformly to a function f on a subset E of $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $(x_n)_{n=1}^\infty \subseteq E$ such that $x_n \to x$ and $x \in E$.

9) (10 points) Let f be a continuous real-valued function on $\mathbb{R}$ with the following properties: $0 \le f(t) \le 1$ and $f(t+2) = f(t)$ for all $t \in \mathbb{R}$ and

$$f(t) = \begin{cases} 0 & 0 \le t \le 1/3 \\ 1 & 2/3 \le t \le 1 \end{cases}.$$

Let $\phi(t) = (x(t), y(t))$ where

$$x(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n-1}t), \qquad y(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n}t).$$

Prove that ϕ is continuous from $\mathbb{R}$ to $\mathbb{R}^2$ where $\mathbb{R}^2$ has the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and that ϕ maps *onto* the unit square $[0,1] \times [0,1]$. This is an example of a *space-filling curve.*

Hint: Each $(x_0, y_0) \in [0,1] \times [0,1]$ has the form

$$x_0 = \sum_{n=1}^\infty \frac{a_{2n-1}}{2^n}, \qquad y_0 = \sum_{n=1}^\infty \frac{a_{2n}}{2^n}$$

with $a_n \in \{0,1\}$ for all $n \in \mathbb{N}$. If

$$t_0 = \sum_{i=1}^{\infty} \frac{2a_i}{3^{i+1}},$$

show that $f(3^k t_0) = a_k$ and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.

10) (10 points) For each positive integer n, define $f_n : [-1,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \le x \le -1/n \\ nx & \text{if } -1/n < x < 1/n \\ 1 & \text{if } 1/n \le x \le 1. \end{cases}$$

Let $C([-1,1])$ denote the set of continuous (in the usual metric), real-valued functions on $[-1,1]$ and for $f, g \in C([-1,1])$, define the metric

$$d(f,g) = \int_{-1}^{1} |f(x) - g(x)|\, dx.$$

Use the sequence $(f_n)_{n=1}^{\infty}$ to show that $(C([-1,1]), d)$ is not a complete metric space.

11) (15 points) Let

$$f(x) = \sum_{k=1}^{n} (x^k - x^{2k}).$$

Show that $(f_n)_{n=1}^{\infty}$ converges pointwise to a function f on $[0,1]$ but that the convergence is not uniform.

12) (15 points) If $x \neq 0$, show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \cos(kx/n) = \frac{\sin(x)}{x}$$

pointwise.