Math 452/552 Problem Set 4

To Be Presented Wednesday, March 20

1) Problems 1-5, 13 in Rudin: 5 points each.

2) Problems 6-12, 14-24, 26-29 in Rudin: 10 points each.

3) (5 points- Exercise 8.1.7) Apply the algorithm in the proof of Theorem 8.1.5 to each half of the interval $[a, b]$ and then explain why this procedure must eventually terminate after some finite number of steps.

4) (5 points each part- Exercise 7.6.17, $(f_n)_{n=1}^{\infty}$ are the functions from Exercise 7.6.16 and C is the Cantor set)

a) Set $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0, 1]$. Explain why $f'(x)$ exists for all $x \notin C$.

b) If $c \in C$, argue that $|f(x)| \leq (x - c)^2$ for all $x \in [0, 1]$. Show how this implies $f'(c) = 0$.

5) (10 points, all or nothing- Exercise 7.6.16, see the book for the definition of $(f_n)_{n=1}^{\infty}$, C is the Cantor set)

a) If $c \in C$, what is $\lim_{n\to\infty} f_n(c)$?

b) Why does $\lim_{n\to\infty} f_n(x)$ exist for $x \notin C$?

6) (5 points- Exercise 8.2.12) If E is a subspace of a metric space (X, d), show that E is nowhere-dense in X if and only if $(\overline{E})^c$ is dense in X.

7) (5 points- Exercise 8.2.14) If E is a nowhere-dense subset of a complete metric space (X, d), what can we say about $(\overline{E})^c$? Now prove the Baire Category Theorem. You may use Theorem 8.2.10.

8) (15 points) Suppose that F and G are differentiable maps of a neighborhood V of a point $x_0 \in \mathbb{R}^n$ into $\mathbb{R}$ and that $F(x_0) = G(x_0)$. Next let $f : V \to \mathbb{R}$ and suppose $F(x) \leq f(x) \leq G(x)$ for all $x \in V$. Prove that f is differentiable at $x = x_0$.

9) (15 points) Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be continuously differentiable (that is, the derivative Df of f exists everywhere and is continuous). Prove or provide a counterexample to the following statement: the set of points where $Df(x)$ has a null space of dimension 2 or greater is closed in $\mathbb{R}^4$.

10) (15 points) Is the set of all invertible 2×2 matrices with real coefficients connected? Here, the metric is inherited from the sup norm on $M_n(\mathbb{R})$.