Math 452/552 Practice Problems for the Final

1) Let $g : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and define functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^1 g(x,y)y^n \, dy$$

for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that the sequence $(f_n)_{n=1}^{\infty}$ has a subsequence which converges uniformly on $[0,1]$.

2) Consider the 1-form F defined on $\mathbb{R}^2 \backslash \{0\}$ by

$$F = \frac{xdy - ydx}{x^2 + y^2}.$$

Evaluate $\int_{\partial C} F$ where C is the unit square $[-1,1] \times [-1,1]$ in $\mathbb{R}^2$, positively oriented.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following property: for every positive integer n and every $x, y \in \mathbb{R}$ such that both $|x| + |y| > n^2$ and $|x - y| < 1/n^2$, we have $|f(x) - f(y)| < 1/n$. Show that f is uniformly continuous.

4) Let $f : [0,1] \to \mathbb{R}^n$ be a continuous function. Show that

$$\left\| \int_0^1 f(t) \, dt \right\|_2 \leq \int_0^1 \|f(t)\|_2 \, dt.$$

5) Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and assume that $Df(x,y)$ is invertible for all $(x,y) \in \mathbb{R}^2$. Assume moreover that, for any compact set $K \subset \mathbb{R}^2$, $f^{-1}(K)$ is compact. Prove that f is onto.

6) Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$g(x,y) = (x^2 + y^2 - |x^2 - y^2|, x^2 + y^2 + |x^2 - y^2|).$$

a) Determine those points $(x_0, y_0) \in \mathbb{R}^2$ where $Dg(x_0, y_0)$ exists and where it does not exist. In both cases, justify your answer. Be sure to analyze $(x_0, y_0) = (0,0)$.

b) Find those points $(x_0, y_0) \in \mathbb{R}^2$ where g locally has a differentiable inverse and where it does not. In both cases, justify your answer.

7) Consider the surface S in $\mathbb{R}^3$ consisting of all points (x, y, z) such that $x^2 + y^2 + z^2 = 1$ and $x \geq 0$, and choose an orientation for S. Calculate the integral $\int_S \omega$ where the 2-form ω is defined by

$$\omega(x, y, z) = x dx \wedge dy + y dy \wedge dz + z dz \wedge dx$$

for $(x, y, z) \in \mathbb{R}^3$.