Math 452/552 Final

Thursday, April 25th

GENERAL RULES: Only your pencil and your brain are permitted for use on this exam. By signing your name to your exam, you assent that you have read and understood these instructions.

1) (20 points) Let f be a continuous function on $[0, 1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\ dx = 0.$$

2) (20 points) Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued continuous functions defined on the interval $[0,1]$ converging *uniformly* to a real-valued function f on $[0,1]$. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points converging to $x \in [0,1]$ with the property that for each $n \in \mathbb{N}$,

$$f_n(x_n) \geq f_n(y)$$

for all $y \in [0,1]$. Prove that $f(x) \geq f(y)$ for all $y \in [0,1]$.

3) (20 points) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be functions with continuous first derivative such that the map $F : (x, y) \to (f, g)$ has Jacobian determinant

$$\det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

identically equal to one.

a) Show that F is an open map, i.e., if $S \subset \mathbb{R}^2$ is open, then $F(S) \subset \mathbb{R}^2$ is open (in both instances, "open" in $\mathbb{R}^2$ is with respect to the metric $d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$).

b) If f is also linear (which implies $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are constant), show that $F(x, y) \neq F(0, 0)$ for any $(x, y) \neq (0, 0)$. *Hint:* for a fixed $(x, y) \in \mathbb{R}^2$, define $h : \mathbb{R} \to \mathbb{R}^2$, $h(t) = F(tx, ty)$. Calculate h'.

BONUS: With F as above and f linear, prove F is 1-1.

4) (20 points) Choose ONE of the following two questions. If you attempt both, I will grade the one you do WORSE on.

Let n be an integer greater than 1 and consider the following statement: if ω is a differential 2-form on $\mathbb{R}^n$ with the property that $\omega \wedge \lambda = 0$ for every differential 1-form λ, then ω must be the zero form. For what n is the above statement true? For what n is it false? Prove your answers.

-OR-

Let ω be a differentiable 1-form on $\mathbb{R}^2$ that satisfies

$$\omega \wedge dx = -d(x^2) \wedge dy$$

and

$$\omega \wedge dy = dx \wedge d(y^2).$$

Let $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = \left(-\sin\left(\frac{\pi t}{2}\right), 4t^3\right)$. Note $\gamma(0) = (0,0)$ and $\gamma(1) = (-1,4)$. Compute

$$\int_\gamma \omega.$$

5) (70 points) Compute $\int_0^1 x \, dx$.