Name:

Math 115 Final

1. WRITE YOUR NAME ON THIS TEST!
2. Except where indicated, merely finding the answer to a problem is not enough to receive full credit; you must show how you arrived at that answer.
3. Unless indicated, DO NOT convert irrational numbers such as $\sqrt{3}$ or π into decimal approximations; just leave them as they are.

1) (10 points, 2 points each) True/False. No justification is necessary.

a) For all continuous functions f and g on $[a, b]$,

$$\int_a^b f(x)g(x)\ dx = \int_a^b f(x)\ dx \cdot \int_a^b g(x)\ dx.$$

b) For all real numbers x and y, $\sqrt{x^2 + y^2} = x + y$.

c) For all polynomials p and all real numbers a, $\lim_{x \to a} p(x) = p(a)$.

d) For all real numbers x, $\sin(\cos(x)) = \cos(x)\sin(x)$.

e) If $f'(c) = 0$, then f has a local maximum or a local minimum at $x = c$.

2) (10 points, 2 points each) Fill-in-the-blank. No justification is necessary.

a) If $f(x) = \int_{-\pi}^{x} \cos^3(t^2)\ dt$, then $f'(x) =$ ______________.

b) $\int_{-4}^{4} |x|\ dx =$______________.

c) The vertical asymptote(s) for the function $f(x) = \dfrac{(x-2)(x+3)}{(3x-5)(x-2)}$ are

at $x =$ ______________.

d) Two continuous antiderivatives of a function f differ by a _____________________.

e) For the integral $\int \dfrac{\sec^2(x)}{(1+\tan(x))^2}\ dx$, a reasonable substitution would

be $u =$ ____________________________.

3) Compute the derivatives of the following functions.

a) (5 points) $f(x) = \dfrac{1}{x^4} + \sqrt{3}x + e^2$

b) (8 points) $g(x) = \sqrt{2x}(3 + \cos x)$

c) (7 points) $h(x) = \dfrac{\tan(\sin x)}{x}$

4) Consider a function f whose domain is all real numbers not equal to ± 9 and whose *derivative* is given by

$$f'(x) = \frac{x - 41}{x^2 - 81}.$$

a) (4 points) Find all critical numbers for f, if any exist.

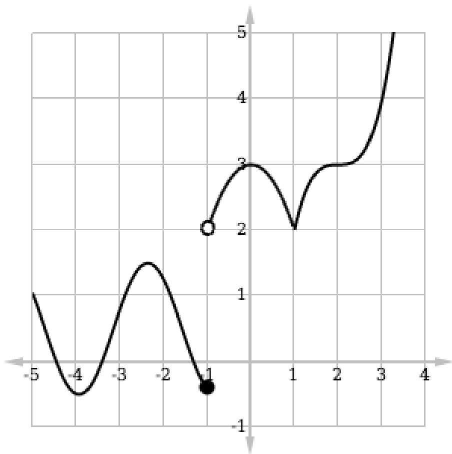
b) (4 points) Determine the intervals where f is increasing or where f is decreasing.

c) (1 point) Record the x-coordinates of the local maxima and minima for f, if any exist.

d) (11.5 points) Find all inflection points for f, if any exist.

e) (3.5 points) Determine the intervals of concavity for f.

5) (16 points) Below is the graph of a function $f(x)$. Use the graph to answer the following.



(a) $\lim_{x \to -1^-} f(x) =$ ____________. If the limit does not exist, explain why.

(b) Find all x values where $f'(x) = 0$

(c) Give all the x values where $f(x)$ is **not** continuous.

(d) Give all the x values where $f(x)$ is **not** differentiable.

(e) On the interval $(1, 2)$ $f(x)$ is
(circle one) INCREASING / DECREASING / NEITHER and
(circle one) CONCAVE UP / CONCAVE DOWN / NEITHER.

For each pair of values, determine if they are equal or if one is larger; that is, fill in the blank with $=$, $<$ or $>$.

(f) $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ _______ 0

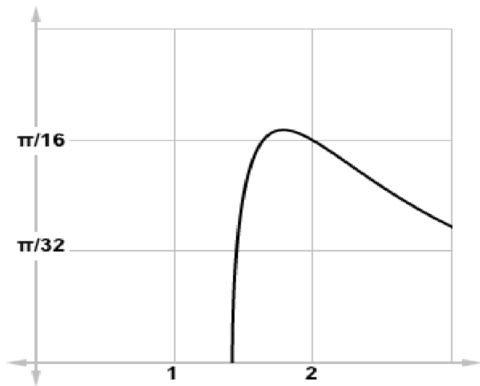
(g) $f'(-2)$ _______ $f'(-3)$

(h) $f''(-4)$ _______ 0

6) Consider the graph of the equation

$$x\cos(x^2y) = \sqrt{2}.$$

See the picture below.



a) (2 points) Draw, on the picture above, the tangent line to the graph at the point $(2, \pi/16)$.

b) (18 points) Find the equation of the tangent line to the graph at the point $(2, \pi/16)$.

7) Let $f(x) = 6x^7 + 4x^5 + 23x + \sqrt{x} - 17$.

a) (6 points) Show that f has a real zero (root) in the interval $[0, 1]$.

b) (9 points) Show that f cannot have more than one real zero.

8) Evaluate the following integrals.

a) (6 points) $\int 5x^{3/4} - 8\sqrt{x} + \pi^6 \; dx$

b) (7 points) $\int_{-1}^{0} \frac{2x+3}{(x^2+3x-1)^2} \; dx$

c) (7 points) $\int_{-13}^{0} \sqrt{169 - x^2} \; dx$

9) Find the value of the limits, if they exist.

a) (3 points) $\lim_{x \to 8} \frac{\sqrt{x+1} - 7}{x - 6}$

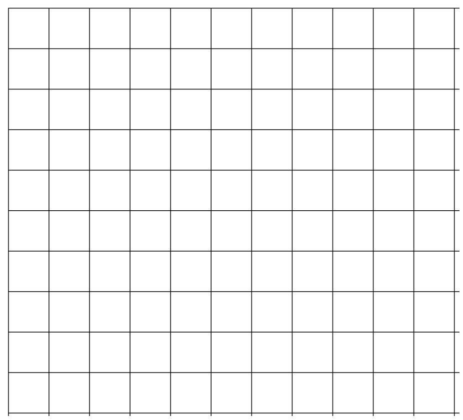
b) (7 points) $\lim_{x \to 8} \frac{\sqrt{x+1} - 3}{x - 8}$

c) (5 points) $\lim_{x \to -\infty} \frac{\sqrt{7x^2 - 3}}{15x}$

10) Consider the region bounded by

$$y = -1 - x^2 \qquad \text{and} \qquad y = -x - 7$$

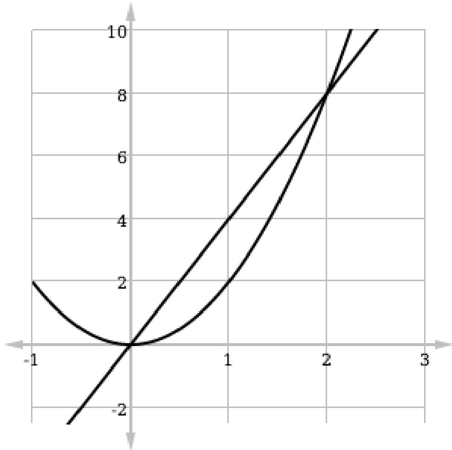
a) Graph the functions and shade the region.



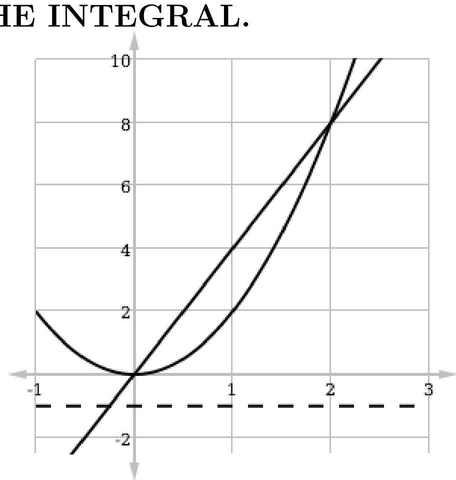
b) Set up a definite integral that represents the area between the curves.

11) Let $\mathcal{R}$ be the region enclosed by $y = 2x^2$ and $y = 4x$ (graphed and shaded below). The curves intersect at $(0,0)$ and $(2,8)$.

a) Set up an integral to compute the volume of the solid obtained by rotating the region $\mathcal{R}$ about the x-axis. **DO NOT EVALUATE THE INTEGRAL.**



b) Set up an integral to compute the volume of the solid obtained by rotating the region $\mathcal{R}$ about the line $y = -1$. **DO NOT EVALUATE THE INTEGRAL.**



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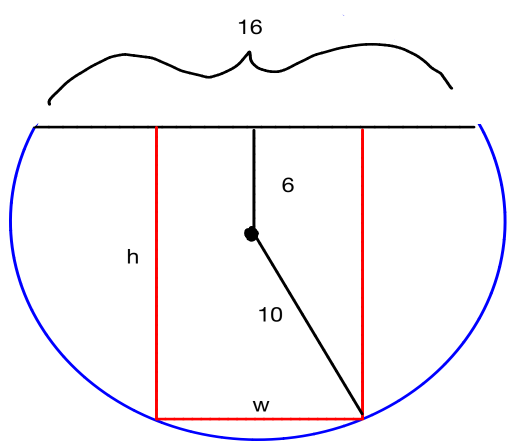
12) Having successfully cloned dinosaurs on Isla Nublar, Dr. Hammond needs to pen them in. He wants a rectangular pen that extends all the way to Isla Nublar's glorious white sand beach and encloses as much area as possible. Isla Nublar has the shape of a circle with a 10 mile radius and a "cap" cut off on the top. See the picture below, all units are in miles.



a) Find a formula for h in terms of w, or vice-versa.

b) Find a one-variable formula for the area of the pen.

c) Determine the dimensions of the pen with maximum area (*Hint:* square the formula you got in b) and maximize that.)

BONUS 1: (10 points) If f is differentiable at $x = a$, where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}.$$

NO credit will be given for evaluating the limit on your favorite examples.

BONUS 2: (10 points) A circle of radius 1 is tangent to both rays determining the graph of $y = 2|x|$. What is the y-coordinate of the center of the circle?