1) a) $x = r\cos\theta = 13\cos\left(-\frac{5\pi}{3}\right) = \frac{13}{2}$

$y = r\sin\theta = 13\sin\left(-\frac{5\pi}{3}\right) = \frac{13\sqrt{3}}{2}$

b) $r = 15$, $\theta = \frac{3\pi}{2}$ or

$r = -15$ $\theta = \frac{\pi}{2}$

2) a) $x'(t) = 3t^2$

$$y'(t) = \frac{1}{-\cos(\pi t)} \cdot \pi \sin(\pi t) + 2t$$

$$y'(t) = -\pi \tan(\pi t) + 2t$$

$x = -1, \quad t^3 = -1, \quad t = -1$

so $x'(-1) = 3$

$y'(-1) = -2$

$$y - 1 = -\frac{2}{3}(x+1)$$

b) parameterize as $x(t)=t$, $y(t)=-\frac{2}{3}(t+1)+1$

$$x'(t)=1, \quad y'(t)=-2/3$$

$$L=\int_0^2 \sqrt{1^2+(-2/3)^2}$$

$$=2\sqrt{13/9}$$

$$=\frac{2\sqrt{13}}{3}$$

3) a) $\frac{dx}{dt} = (.5)(20) - \frac{x(t)}{1500} \cdot 20$

$\frac{dx}{dt} = 10 - \frac{x(t)}{75}$

b) $\frac{dx}{dt} = \frac{750 - x(t)}{75}$

$\frac{1}{750 - x(t)} \frac{dx}{dt} = \frac{1}{75}$

$\int \frac{1}{750 - x(t)} \frac{dx}{dt} \, dt = \int \frac{1}{75} \, dt$

$u = 750 - x(t)$

$du = -\frac{dx}{dt} \, dt$

$$-\int \frac{1}{v}\,dv = \int \frac{1}{75}\,dt$$

$$-\ln|v| = \frac{t}{75} + C$$

$$-\ln|750 - x(t)| = \frac{t}{75} + C$$

$x(t) \leq 750$, so remove absolute values

$$-\ln(750 - x(t)) = \frac{t}{75} + C$$

$$x(0) = 2$$

$$-\ln(748) = C$$

$$-\ln(750 - x(t)) = \frac{t}{75} - \ln(748)$$

$$\ln(750 - x(t)) = -\frac{t}{75} + \ln(748)$$

$$750 - x(t) = e^{-\frac{t}{75} + \ln(748)}$$

$$= e^{-\frac{t}{75}} \cdot 748$$

$$x(t) = 750 - e^{-\frac{t}{75}} \cdot 748$$

c) $x(15) = 750 - 748\, e^{-1/5}$

$$\approx 137.5824 \text{ kg}.$$

4) a) $\mathcal{L}\{t^2 \cdot a^t\}$

$$= \int_0^\infty t^2 a^t e^{-\omega t}\, dt$$

$$= \lim_{x \to \infty} \int_0^x t^2 a^t e^{-\omega t}\, dt$$

$$\int_0^x t^2 a^t e^{-\omega t}\, dt$$

$$= \int_0^x t^2 e^{\ln(a^t)} e^{-\omega t}\, dt$$

$$= \int_0^x t^2 e^{(\ln(a) - \omega)t}\, dt$$

Tabular

u	dv
t^2	$e^{(\ln(a)-w)t}$
$2t$	$e^{(\ln(a)-w)t}/(\ln(a)-w)$
2	$e^{(\ln(a)-w)t}/(\ln(a)-w)^2$
0	$e^{(\ln(a)-w)t}/(\ln(a)-w)^3$

$$\int_0^x t^2 e^{(\ln(a)-w)t}\,dt$$

$$= \left(\frac{e^{(\ln(a)-w)t}}{(\ln(a)-w)}\left(t^2 - \frac{2t}{\ln(a)-w} + \frac{2}{(\ln(a)-w)^2}\right)\right)\Bigg|_0^x$$

$$= \frac{-2}{(\ln(a)-w)^3} + \frac{e^{(\ln(a)-w)x}}{(\ln(a)-w)}\left(x^2 - \frac{2x}{\ln(a)-w} + \frac{2}{(\ln(a)-w)^2}\right)$$

If $\omega > \ln(a)$, then $\ln(a)-\omega < 0$, so

$$\lim_{x\to\infty} \frac{e^{(\ln(a)-\omega)x}}{\ln(a)-\omega}\left(x^2 - \frac{2x}{(\ln(a)-\omega)} + \frac{2}{(\ln(a)-\omega)^2}\right)$$

$$= \frac{1}{\ln(a)-\omega} \lim_{x\to\infty} \frac{x^2 - \frac{2x}{(\ln(a)-\omega)} + \frac{2}{(\ln(a)-\omega)^2}}{e^{(\omega-\ln(a))x}}$$

$$\overset{\text{l'H}}{=} \frac{1}{\ln(a)-\omega} \lim_{x\to\infty} \frac{2x - \frac{2}{\ln(a)-\omega}}{(\omega-\ln(a))e^{(\omega-\ln(a))x}}$$

$$\overset{\text{l'H}}{=} \frac{1}{\ln(a)-\omega} \lim_{x\to\infty} \frac{2}{(\omega-\ln(a))^2 e^{(\omega-\ln(a))x}} = 0$$

This is sufficient, but they could write more. Then the Laplace Transform is

$$\frac{-2}{(\ln(a)-\omega)^3} \quad \text{for}$$

b) $\ln(a) < \omega$. If $\omega > \ln(a)$, the limit diverges.

If $\omega = \ln(a)$, the integral becomes

$$\int_0^x t^2\,dt = \frac{t^3}{3}\Big|_0^x$$

$$= \frac{x^3}{3}$$

$$\lim_{x\to\infty} \frac{x^3}{3} = \infty$$

5) a) $$\frac{a_{n+1}}{a_n} = \frac{(n+1)x^n}{2^n} \cdot \frac{2^{n-1}}{n x^{n-1}}$$

$$= \frac{n+1}{n} \cdot \frac{1}{2} \cdot x$$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x| \cdot \frac{n+1}{2n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \cdot \frac{n+1}{2n}$$

$$= \frac{|x|}{2} \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \frac{|x|}{2} < 1$$

$$|x| < 2$$

$$R = 2$$

b) $$f(x)=\sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{2^{n-1}} = \sum_{n=0}^{\infty}(n+1)\left(\frac{x}{2}\right)^n$$

$$y-(2-x)y'=0$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty} n\,x^{n-1}$$

$$\frac{1}{\left(1-\frac{x}{2}\right)^2}=\sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1}=f(x)$$

$$f(x)=\frac{4}{(2-x)^2}$$

$$f'(x) = 8(2-x)^{-3}$$

$$= \frac{8}{(2-x)^3}$$

$$f(x) = \frac{4}{(2-x)^2}$$

$$2f(x) - 2f'(x) + xf'(x)$$

$$= \frac{8}{(2-x)^2} - \cancel{(2-x)} \cdot \frac{8}{(2-x)^{\cancel{3}}}$$

$$= 0 \checkmark$$

c) $f(1) = \frac{4}{(2-1)^2} = 4$

6) Residue Theorem