Final Winter '22

1) a) (i) $-5, -3$

(ii) $10, 6$

(iii) $-6, -2$

b) $-1 < x < 1$

c) False

d) False

e) $x = 2\sec(\theta)$

2) a) $x = r\cos(\theta) = -10\cos(-7\pi/3) = -5$

$y = r\sin(\theta) = -10\sin(-7\pi/3) = 5\sqrt{3}$

b) $r = \sqrt{x^2 + y^2}$

$= \sqrt{(18)^2 \cdot 3 + 18^2}$

$= 36$

$\theta = \arctan\left(\frac{y}{x}\right) + \pi$

$= \arctan\left(\frac{-18}{-18\sqrt{3}}\right) + \pi$

$= \arctan\left(\frac{1}{\sqrt{3}}\right) + \pi$

$= \frac{\pi}{6} + \pi = \frac{7\pi}{6}$

$(36, 7\pi/6), \quad (36, \frac{7\pi}{6} + 2\pi)$

3

$$x(t) = \ln(25) - \ln(t^2)$$
$$= \ln(25) - 2\ln(t)$$

$$x'(t) = -\frac{2}{t}$$

$$y'(t) = \frac{1}{5} \quad \frac{1}{1 + (t/5)^2}$$

$$\frac{\pi}{4} = \arctan(t/5)$$

$$1 = t/5 \quad , t = 5$$

$$x'(5) = -2/5$$

$$y'(5) = \frac{1}{10}$$

$$\frac{y'(5)}{x'(5)} = -1/4$$

$$y - \pi/4 = -1/4x$$

4

a) $14x - 7 = 0$

$x = 1/2$

b) $a_n = \dfrac{(14x-7)^n}{42^n}$

$a_{n+1} = \dfrac{(14x-7)^{n+1}}{42^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{(14x-7)^{\cancel{n}+1}}{42^{\cancel{n}+1}} \cdot \frac{\cancel{42^n}}{\cancel{(14x-7)^n}}$$

$$= \frac{14x-7}{42}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{14x-7}{42}\right| < 1$$

$$-1 < \frac{14x-7}{42} < 1$$

$$-42 < |4x - 7| < 42$$

$$-3 < x - \tfrac{1}{2} < 3$$

$$R = 3$$

— or —

geometric

5 a) $x'(t) = 2t\cos(t^2)$

$y'(t) = 2t\sin(t^2)$

$$L = \int_0^{\sqrt{\pi/2}} \sqrt{(2t\cos(t^2))^2 + (2t\sin(t^2))^2}\, dt$$

b) $$L = \int_0^{\sqrt{\pi/2}} \sqrt{4t^2\cos^2(t^2) + 4t^2\sin^2(t^2)}\, dt$$

$$= \int_0^{\sqrt{\pi/2}} \sqrt{4t^2\,(\cos^2(t^2) + \sin^2(t^2)}\, dt$$

$$= \int_0^{\sqrt{\pi/2}} 2t\, dt = t^2 \Big|_0^{\sqrt{\pi/2}}$$

$$= \pi/2$$

c

$$3x^2 = x^2 e^x$$

$$3x^2 - x^2 e^x = 0$$

$$x^2(3 - e^x) = 0$$

$$x = 0 \quad \text{or} \quad 3 - e^x = 0$$

$$x = \ln(3)$$

$$\int_0^{\ln(3)} 3x^2 - x^2 e^x \, dx$$

$$= \int_0^{\ln(3)} 3x^2 \, dx - \int_0^{\ln(3)} x^2 e^x \, dx$$

$$= x^3 \Big|_0^{\ln(3)}$$

$$= (\ln(3))^3$$

$$\int_0^{\ln(3)} x^2 e^x \, dx =$$

u		dv
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

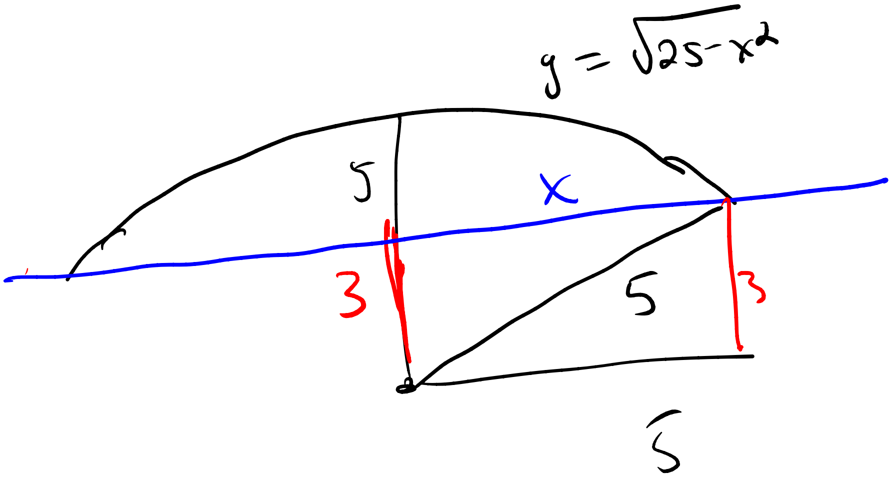
$$= x^2 e^x - 2xe^x + 2e^x \Big|_0^{\ln(3)}$$

$$= 3\left(\ln(3)^2 - 2\ln(3) + 2\right) - 2$$

Total area:

$$(\ln(3))^3 - 3\left(\ln(3)^2 - 2\ln(3)\right) - 4$$

7) a)



$$5^2 = 3^2 + x^2, \quad x = 4$$

$$\text{Volume} = 2\left(\int_0^4 \pi\left(\sqrt{25-x^2}\right)^2 dx - \int_0^4 9\pi \, dx\right)$$

b) $$\text{Volume} = 2\pi \int_0^4 25 - x^2 \quad - 36\pi$$

$$= 2\pi\left(25x - \frac{x^3}{3}\Big|_0^4\right) - 36\pi$$

$$= 2\pi\left(\left(\frac{236}{3}\right) - 36\pi\right) = \frac{256\pi}{3}$$

8 a) $\mathcal{L}[f](4) = \int_0^\infty f(t) e^{-4t} \, dt$

$$= \int_0^\infty e^{2t} e^{-4t} \, dt$$

$$= \int_0^\infty e^{-2t} \, dt$$

$$= \lim_{a \to \infty} \int_0^a e^{-2t} \, dt$$

$$= \lim_{a \to \infty} \frac{e^{-2t}}{-2} \Big|_0^a$$

$$= \lim_{a \to \infty} \frac{e^{-2a}}{-2} + \frac{1}{2}$$

$$= 1/2$$

b) No!

$$\mathcal{L}(f)(2) = \int_0^\infty e^{2t} e^{-2t} \, dt$$

$$= \int_0^\infty 1 \, dt$$

$$= \infty \quad +1 \quad \text{divergent}$$

9

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n!}$$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= x^3 e^{-x^2}$$

10 a) $\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k$

where $S_k = \sum_{n=1}^{k} a_n$

b) $\frac{1}{n^2+13n+42} = \frac{1}{(n+7)(n+6)}$

$= \frac{A}{n+7} + \frac{B}{n+6}$

$1 = A(n+6) + B(n+7)$

$n = -6$, $n = -7$,

$A = 1$ $B = -1$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2+13n+42}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{n+7}-\frac{1}{n+6}\right)$$

$$S_1=\frac{1}{8}-\frac{1}{7}$$

$$S_2=\cancel{\frac{1}{8}}-\frac{1}{7}+\frac{1}{9}-\cancel{1/8}=1/9-1/7$$

$$S_3=\cancel{1/9}-1/7+1/10-\cancel{1/9}=1/10-1/7$$

$$S_4=\cancel{1/10}-1/7+1/11-\cancel{1/10}=1/11-1/7$$

$$S_k=\frac{1}{k+7}-\frac{1}{7}$$

$$\lim_{k\to\infty}S_k=-1/7$$

11 a) $\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$

$= 6\ L/\text{min} \cdot .02\ \text{kg}/L - 6\ L/\text{min}\ \frac{x(t)}{1000}\ \text{kg}/L$

$= 6\left(.02 - \frac{x(t)}{1000}\right)$

b) $\frac{dx}{dt} = 6\left(.02 - \frac{x(t)}{1000}\right) = 6\left(\frac{20 - x(t)}{1000}\right)$

$\frac{1}{(20 - 6x(t))} \frac{dx}{dt} = \frac{1}{1000}$

$\int \frac{1}{(20-6x(t))} \frac{dx}{dt}\, dt = \int \frac{1}{1000}\, dt$

$= \frac{t}{1000} + C$

$$\int \frac{1}{120 - 6x(t)} \frac{dx}{dt}\, dt$$

$$u = 120 - 6x(t)$$

$$du = -6 \frac{dx}{dt}\, dt$$

$$-\frac{1}{6} \int \frac{1}{u}\, du$$

$$= -\frac{\ln(u)}{6}$$

$$= -\frac{\ln(120 - 6x(t))}{6}$$

$$\ln(120 - 6x(t)) = -\frac{3t}{500} + C$$

$$e^{\ln(120 - 6x(t))} = e^{-\frac{3t}{500} + C}$$

$$120 - 6x(t) = e^{-\frac{3t}{500} + C}$$

$x(0) = 8$, so

$$120 - 48 = e^{C}$$

$$C = \ln(72)$$

$$6\,x(t) = 120 - e^{-\frac{3t}{500} + \ln(72)}$$

$$x(t) = 20 - 12\,e^{-3t/500}$$

c) $x(15) = 20 - 12\,e^{-3t/500}$

12 a) $$\frac{(-74)^{n+2}}{4^{3n}} = \frac{(-74)^2(-74)^n}{64^n}$$

$$= (-74)^2\left(\frac{-74}{64}\right)^n$$

$$\left|\frac{-74}{64}\right| = \frac{74}{64} > 1$$

So the series diverges (could also use ratio test)

5) $$\lim_{n\to\infty}\left(\frac{n^3}{n^2-8}\right)^{5n^2}$$

$$= \lim_{n\to\infty}\left(\frac{n^2-8}{n^2}\right)^{-5n^2}$$

$$= \lim_{n\to\infty}\left(1-\frac{8}{n^2}\right)^{-5n^2}$$

$$= \lim_{n\to\infty}\left(\left(1-\frac{8}{n^2}\right)^{n^2}\right)^{-5}$$

$$\left(1-\frac{8}{n^2}\right)^{n^3} = e^{\ln\left(\left(1-\frac{8}{n^2}\right)^{n^2}\right)}$$

$$= e^{n^2\ln\left(1-\frac{8}{n^2}\right)}$$

$$= e^{\frac{\ln(1-8/n^2)}{1/n^2}}$$

$$\lim_{n\to\infty} \frac{\ln(1-8/n^2)}{1/n^2} \qquad \left(\frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1-8/n^2} \cdot -8 \cdot \cancel{-\frac{2}{n^3}}}{\cancel{-2/n^3}}$$

$$= \lim_{n\to\infty} \frac{-8}{1-8/n^2} = -8$$

$$\lim_{n\to\infty} \left(1-\frac{8}{n^2}\right)^{n^2}$$

$$= e^{-8}$$

Final answer: e^{40}