Final W '23

1) a) (i) $-5.5, -4.5$

(ii) $0, -7$

(iii) $-4, -6$

b) $-1 < x < 1$

c) F

d) T

e) $2\sin(\theta)$

2) a) $x = r\cos\theta = 18\cos(7\pi/6)$

$= -9\sqrt{3}$

$y = r\sin\theta = 18\sin(7\pi/6)$

$= -9$

$(-9\sqrt{3}, -9)$

b) $r = \sqrt{x^2+y^2} = \sqrt{4^2+(-4)^2}$

$= \sqrt{32}$

$= 4\sqrt{2}$

in 2^{nd} quadrant, so

$\theta = \arctan(\frac{y}{x}) + \pi$

$= \arctan(\frac{4}{-4}) + \pi$

$= \arctan(-1) + \pi = 3\pi/4$

$(4\sqrt{2}, 3\pi/4)$

3)

$$x'(t) = \arctan(t-1) + \frac{t}{(t-1)^2+1}$$

$$y'(t) = e^{t^2-1} \cdot 2t$$

$$x'(1) = 1$$

$$y'(1) = 2$$

$$x(1) = 0$$

$$y(1) = 1$$

$$y - 1 = 2x$$

$$y = 2x + 1$$

4) a) $c = 2$

b) $a_n = \dfrac{(x-2)^n}{n^2+1}$

$a_{n+1} = \dfrac{(x-2)^{n+1}}{(n+1)^2+1}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \dfrac{n^2+1}{(x-2)^n}$

$= \dfrac{(x-2)^{n+1}}{(x-2)^n} \cdot \dfrac{n^2+1}{(n+1)^2+1}$

$= (x-2) \cdot \dfrac{n^2+1}{(n+1)^2+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = |x-2| \cdot \frac{n^2+1}{(n+1)^2+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2| \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}$$

$$\overset{\text{l'H}}{=} |x-2| \lim_{n\to\infty}\frac{2n}{2(n+1)}$$

$$\overset{\text{l'H}}{=} |x-2| \lim_{n\to\infty} 1 = |x-2|$$

$|x-2| < 1$ $\quad$ $\Big($ $-1 < x-2 < 1$

$\qquad\qquad\qquad\qquad 1 < x < 3$

$R = 1$ $\qquad\qquad R = \frac{3-1}{2} = 1$ $\Big)$

5)

a) $x(t) = \ln(6) + \ln(t^2)$

$= \ln(6) + 2\ln(t)$

$x'(t) = \frac{2}{t}$

$y(t) = \ln(3) - \ln(t)$

$y'(t) = -\frac{1}{t}$

$$L = \int_1^e \sqrt{\left(\frac{2}{t}\right)^2 + \left(-\frac{1}{t}\right)^2}\, dt$$

b) $L = \int_1^e \sqrt{\frac{4}{t^2} + \frac{1}{t^2}} \, dt$

$= \int_1^e \sqrt{\frac{5}{t^2}} \, dt$

$= \int_1^e \sqrt{5} \cdot \frac{1}{t} \, dt$

$= \sqrt{5} \ln(t) \Big|_1^e$

$= \sqrt{5}$

6)

$$\frac{x}{-2} = t, \quad \text{so}$$

$$y = -\left(-\frac{x}{2}\right)^2 + 1 = -\frac{x^2}{4} + 1$$

Intersections:

$$0 = -\frac{x^2}{4} + 1$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 4, \qquad x = \pm 2$$

$$\int_{-2}^{2} -\frac{x^2}{4} + 1 \, dx$$

$$= \left(-\frac{x^3}{12} + x\right)\Big|_{-2}^{2} = 8/3$$

7) Look at grading scale

8) $\mathcal{L}\{f\}(6)$

$$= \int_0^\infty (2-3t)e^{-6t}\,dt$$

$$= \lim_{x\to\infty} \int_0^x (2-3t)e^{-6t}\,dt$$

u		dv
		e^{-6t}
$2-3t$	$+$	$e^{-6t}/-6$
-3	$-$	$e^{-6t}/36$
0		

$$\lim_{x\to\infty}\left(-\left(\frac{(2-3t)e^{-6t}}{6} + \frac{3e^{-6t}}{36}\right)\right)\Bigg|_0^x$$

$$= \lim_{x\to\infty}\left(\frac{1}{12e^{6x}} - \frac{(2-3x)}{6e^{6x}}\right) + \frac{1}{4}$$

$$\lim_{x\to\infty} \frac{2-3x}{6e^{6x}}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{-3}{36e^{6x}} = \lim_{x\to\infty} \frac{-1}{12e^{6x}} = 0$$

so $$\lim_{x\to\infty} \frac{1}{12e^{6x}} - \frac{(2-3x)}{6e^{6x}} = 0$$

and

$$\mathcal{L}\{f\}(6) = \frac{1}{4}$$

a) $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+7}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2+5}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^5}{2n+1} x^{4n+2}$$

$$= x^5 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2(2n+1)}}{2n+1}$$

$$= x^5 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1}$$

$$= x^5 \arctan(x^2)$$

10)

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+1)$$

$n=-2$, $\quad$ $n=-1$

$B=-1$ $\quad$ $A=1$

$$\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$12 \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = 12 \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

Partial sums

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - 1/4 = 1/2 - 1/4$$

$$S_3 = \frac{1}{2} - 1/4 + 1/4 - 1/5 = 1/2 - 1/5$$

$$S_4 = 1/2 - 1/5 + 1/5 - 1/6 = 1/2 - 1/6$$

$$S_n = 1/2 - \frac{1}{n+2}$$

$$\lim_{n \to \infty} S_n = 1/2 - \lim_{k \to \infty} \frac{1}{n+2} = 1/2$$

11) a) $S(0) = 10$ kg

b) $$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 20 \cdot .05 - \frac{S(t) \cdot 20}{1500}$$

$$\frac{dS}{dt} = 1 - \frac{S(t)}{75}$$

c) $$\frac{dS}{dt} = -\frac{1}{75}\left(S(t) - 75\right)$$

$$\frac{1}{S(t)-75}\frac{dS}{dt} = -1/75$$

$$\int \frac{1}{S(t)-75}\frac{dS}{dt}\,dt = \int -1/75\,dt$$

$$= -\frac{t}{75} + C$$