Announcements

1) Quiz Thursday over 11.6 & 11.8

2) Exam next Thursday

Radius and Interval of Convergence

Given a power series that is shifted

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

the series can coverge in one of three ways:

1) Only $x = c$

2) A finite interval centered at $x = c$

3) All real numbers

So the series converges in an interval called the interval of convergence

Half the length of the interval of convergence is called the radius of convergence.

The ratio test almost always can find the radius of convergence, but can miss out a bit on the interval in case 2).

Example 1: Find the radius of convergence

of $$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+2}$$

n^{th} term: $$\frac{3^n (x-2)^n}{n+2}$$

$(n+1)^{st}$ term: $$\frac{3^{n+1} (x-2)^{n+1}}{n+3}$$

1) Divide $$\frac{(n+1)^{st}}{n^{th}} = \frac{3^{n+1} (x-2)^{n+1}}{n+3} \cdot \frac{n+2}{3^n (x-2)^n}$$

2) Collect like terms: $$\frac{3^{n+1}}{3^n} \cdot \frac{(x-2)^{n+1}}{(x-2)^n} \cdot \frac{n+2}{n+3}$$

2) Collect like terms: $\frac{3^{\cancel{n}+1}}{\cancel{3^n}} \frac{(x-2)^{\cancel{n}+1}}{\cancel{(x-2)^n}} \frac{n+2}{n+3}$

$$= 3 \, (x-2) \, \frac{n+2}{n+3}$$

3) Take absolute values

$$\left| 3 \, (x-2) \, \frac{n+2}{n+3} \right|$$

$$= |3| \, |x-2| \, \left| \frac{n+2}{n+3} \right|$$

$$= 3 \, |x-2| \, \frac{n+2}{n+3}$$

4) Take limit as $n \to \infty$

$$\lim_{n \to \infty} 3 \, |x-2| \, \frac{n+2}{n+3}$$

4) Take limit as $n \to \infty$

$$\lim_{n\to\infty} 3|x-2|^{\frac{n+2}{n+3}}$$

$$= 3|x-2|^{\underbrace{\lim_{n\to\infty}\frac{n+2}{n+3}}_{=1}}$$

$$= 3|x-2|$$

5) Set less than one and solve

$$3|x-2| < 1$$

$$|x-2| < \frac{1}{3} \quad \text{means}$$

$$-\frac{1}{3} < x-2 < \frac{1}{3}$$

$$-\frac{1}{3} < x-2 < \frac{1}{3}$$

$$+2 \qquad +2 \qquad +2$$

$$\frac{5}{3} < x < \frac{7}{3}$$

Ratio test says:

- series converges when $\frac{5}{3} < x < \frac{7}{3}$
- series diverges when $x < \frac{5}{3}$ or $x > \frac{7}{3}$
- when $x = \frac{5}{3}$ or $x = \frac{7}{3}$, the test tells you nothing

$$\text{Radius} = \frac{1}{2}(\text{length of interval})$$

$$= \frac{1}{2}\cdot\left(\frac{2}{3}\right) = \boxed{\frac{1}{3}}$$

Example 2: Find the radius of convergence for

$$g(x) = \sum_{n=1}^{\infty} (2n)!\,(x-6)^n$$

Definitely converges at $x=6$ (center).

Ratio test

n^{th} term: $(2n)!\,(x-6)^n$

$(n+1)^{st}$ term: $(2(n+1))!\,(x-6)^{n+1}$

$$= (2n+2)!\,(x-6)^{n+1}$$

1) Divide $(n+1)^{st}$ term by n^{th} term:

$$\frac{(2n+2)!\,(x-6)^{n+1}}{(2n)!\,(x-6)^{n}}$$

2) Group like terms

$$\frac{(2n+2)!}{(2n)!}\;\frac{(x-6)^{\cancel{n}+1}}{\cancel{(x-6)^{n}}}$$

$$= \frac{(2n+2)!}{(2n)!}\,(x-6)$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n)!}\,(x-6)}{\cancel{(2n)!}}$$

$$= (2n+2)(2n+1)(x-6)$$

3) Take absolute values

$$|(2n+2)(2n+1)(x-6)|$$

$$= |2n+2|\,|2n+1|\,|x-6|$$

$$= (2n+2)(2n+1)\,|x-6|$$

4) Take limit as $n \to \infty$

$$\lim_{n\to\infty} (2n+2)(2n+1)|x-6| = \infty$$

except when $x=6$.

Then $(2n+2)(2n+1)(6-6)$

$= (2n+2)(2n+1) \cdot 0 = 0$,

so the limit is zero.

5) Set less than one

But the limit is only less than one if $x = 6$.

Then the interval of convergence is just $x = 6$.

$$\text{Radius} = \frac{1}{2}(\text{length of interval})$$

$$= \frac{1}{2} \cdot 0 = \boxed{0}.$$

Power Series Calculus

(Section 11.9)

We would like to use power series to solve differential equations.

To do that, we need to know how to take derivatives!

Given $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

with nonzero radius of convergence,

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

$$f'(x) = 0 + a_1 + a_2 \cdot 2(x-c) + a_3 \cdot 3(x-c)^2 + \cdots$$

Example 3: $y' = y$

Show $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies this equation.

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$f'(x) = 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \frac{5x^4}{120} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$= f(x)$$

This says $f(x) = ce^x$ for some constant c. To find c, set $x=0$.

$$1 = f(0) = c\,e^{0} = c$$

So $\boxed{f(x) = e^{x}}$