Announcements

1) Exam Thursday

Taylor and Maclaurin Series

(section 11.10)

For a given, infinitely differentiable function f at $c=0$, the Maclaurin series for f is the power series

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

where $f^{(n)}(x) = n^{th}$ derivative of f

$$f^{(0)}(x) = f(x)$$

Suppose I know

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \text{ nonzero radius of convergence}$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(0) = a_0$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$$

$$f''(0) = 2a_2$$

$$f'''(x) = 6a_3 + 24a_4 x + \cdots$$

$$f'''(0) = 6a_3$$

Solving for the a_n's,

$$a_0 = f(0) = \frac{f(0)}{0!}$$

$$a_1 = f'(0) = \frac{f'(0)}{1!}$$

$$a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

$$a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

What this calculation says:

If we can write f as a power series centered at $c = 0$ with nonzero radius of convergence, then the power series must be equal to the MacLaurin series of f.

Example 1: Check that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

satisfies the differential equation

$$y'' + y = 0$$ ✓

$y = \sin(x)$ also satisfies this differential equation, as does $y = \cos(x)$.

In differential equations, you will learn: any solution to $y''+y=0$ is of the form

$$y(x) = a\cos(x) + b\sin(x)$$

where a, b are some numbers.

This means

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = a\cos(x) + b\sin(x)$$

Find a and b!

First, plug in $x=0$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = a\cos(x) + b\sin(x)$$

Find a and b!

First, plug in $x=0$.

$$1 = a\cos(0) + b\sin(0)$$

$$= a$$

So $a=1$.

To find b, observe that

$f(x) = f(-x)$ since all powers

in the series are even.

This says

$$f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right).$$

But

$$f(x) = \cos(x) + b\sin(x), \text{ so}$$

$$b = \cos\left(\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}\right)$$

$$= \cos\left(\frac{-\pi}{2}\right) + b\sin\left(-\frac{\pi}{2}\right)$$

$$= -b$$

So $b = -b$, and $b = 0$.

Then $f(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)}$!

By differentiating

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \text{we get}$$

$$\sin(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$$2n-1 = 2n-2+1$$
$$= 2(n-1)+1$$

$$m = n-1$$

Taylor series are of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Where c might not be zero.

Only do this when either f or its derivatives are not defined at $c=0$

(i.e. $\ln(x)$, $\sqrt{x}$)