Announcements

1) Quiz Thursday over
   10.1, 10.2

2) Exam next Thursday over
   10.1 – 10.4, 8.1, 8.2

3) Halloween tomorrow
   costumes = candy

Recall: tangent lines to parametric curves

If $f(t) = \langle x(t), y(t) \rangle$, the equation of the tangent line to the graph of f at $t = t_0$ is

$$\frac{y'(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

Example 1: Find the tangent line

to $g(t) = \langle \arctan(t), \cos(\pi t) \rangle$

at the point $\left(\frac{\pi}{4}, -1\right)$

To find t_0, determine the point as

$$x(t_0) = \frac{\pi}{4} = \arctan(t_0)$$

$$y(t_0) = -1 = \cos(\pi t_0)$$

Use $\arctan(t_0) = \frac{\pi}{4}$,

take tangent of both sides.

$$\tan\left(\frac{\pi}{4}\right) = \tan(\arctan(t_0)) = t_0$$

$$\boxed{t_0 = 1}$$

$$\boxed{t_0 = 1}$$

$$x(t) = \arctan(t)$$

$$y(t) = \cos(\pi t)$$

$$x'(t) = \frac{1}{1+t^2}$$

$$y'(t) = -\sin(\pi t)\cdot\pi$$

$$x'(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$y'(1) = -\sin(\pi)\cdot\pi = 0$$

$$x(1) = \frac{\pi}{4}, \quad y(1) = -1 \quad \text{(given)}$$

$$x'(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$y'(1) = -\sin(\pi)\cdot\pi = 0$$

$$x(1) = \frac{\pi}{4}, \quad y(1) = -1$$ (given)

Equation of line:

$$\frac{y'(1)}{x'(1)} = \frac{y - y(1)}{x - x(1)}$$

$$\frac{0}{(1/2)} = \frac{y+1}{x - \frac{\pi}{4}}$$

$$y + 1 = 0,$$

$$\boxed{y = -1}$$

horizontal line

Vertical / Horizontal Tangents

- When $y'(t_0) = 0$ (and $x'(t_0) \neq 0$) the tangent line is horizontal.

- When $x'(t_0) = 0$ (and $y'(t_0) \neq 0$), the tangent line is vertical.

Arclength

(Section 10.2)

Motivating example:

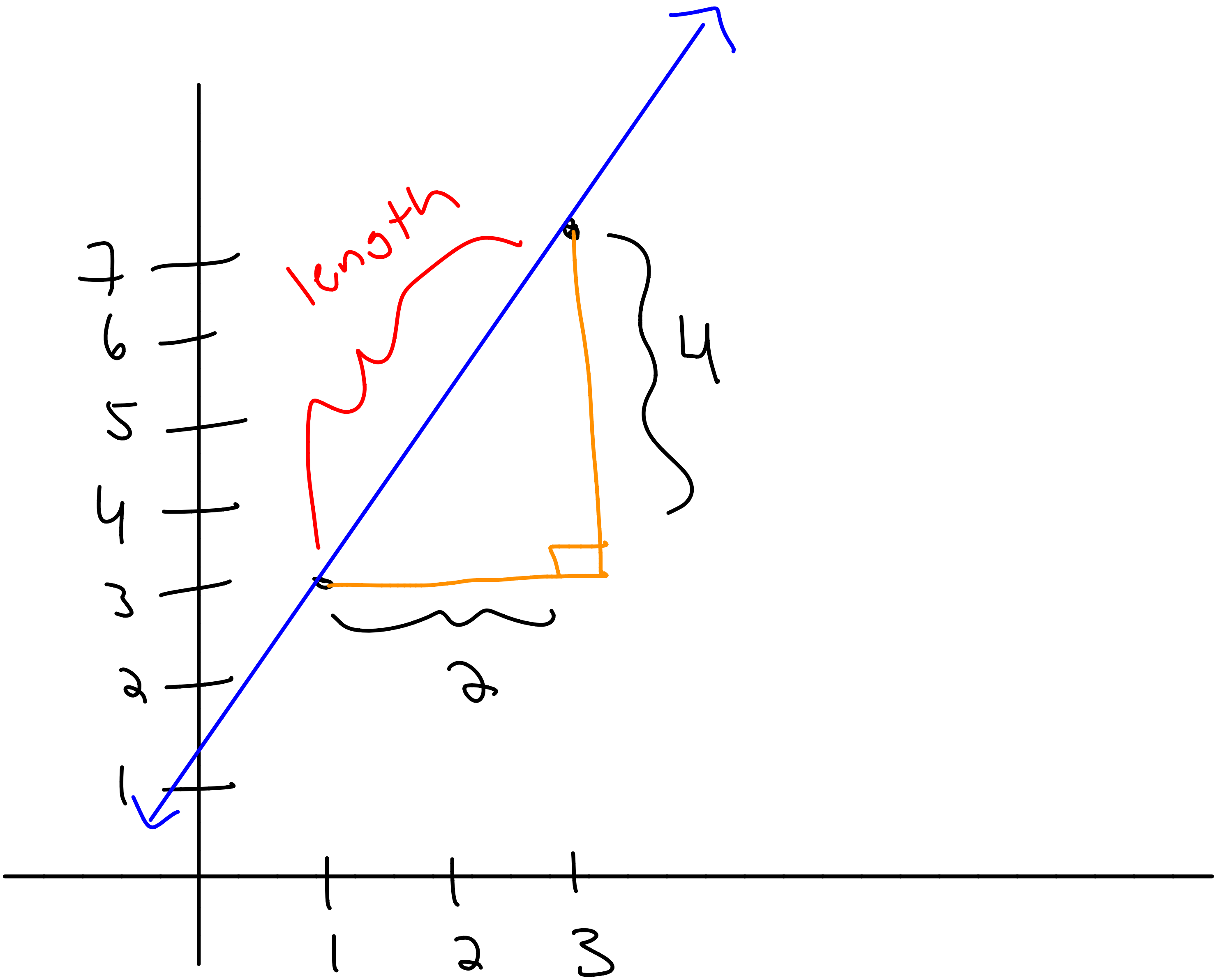
Find the length of

$y = 2x+1$ from $x=1$ to $x=3$

Find the length of

$y = 2x + 1$ from $x = 1$ to $x = 3$

Picture



Use Pythagorean Theorem:

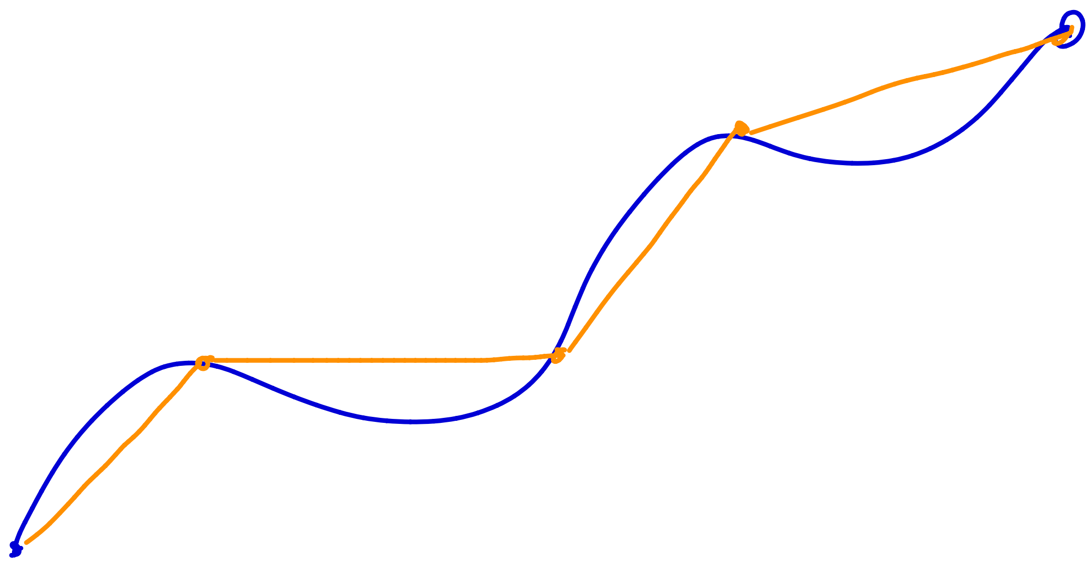
$$(\text{length})^2 = 2^2 + 4^2$$

$$= 20$$

$$\text{length} = \sqrt{20} = 2\sqrt{5}$$

For a general curve:

approximate by line segments



Take smaller and smaller segments, add up, take limit as number of segments goes to infinity (all lengths go to zero)

Formula

The arclength L for a parametric curve $f(t) = \langle x(t), y(t) \rangle$ from $t=a$ to $t=b$ is

$$\boxed{L = \int_a^b \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}}_{\text{distance formula}} \, dt}$$

Need x, y differentiable, x', y' continuous.

Don't forget the derivatives!

Example 1:

Find the arclength of

$$f(t) = \langle e^{3t} + e^{-3t},\ 24 - 6t \rangle$$

from $t = 0$ to $t = \ln(8)$

Formula: $$L = \int_0^{\ln(8)} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$x(t) = e^{3t} + e^{-3t}$$

$$x'(t) = 3e^{3t} - 3e^{-3t}$$

$$y(t) = 24 - 6t$$

$$y'(t) = -6$$

$$x'(t) = 3e^{3t} - 3e^{-3t}$$

$$y'(t) = -6$$

$$(x'(t))^2 = 9e^{6t} + 9e^{-6t} - 18$$

$$+ \qquad\qquad +$$

$$(y'(t))^2 = 36$$

$$= 9e^{6t} + 9e^{-6t} + 18$$

$$= \boxed{\left(3e^{3t} + 3e^{-3t}\right)^2}$$

$$(x'(t))^2 + (y'(t))^2 = (3e^{3t} + 3e^{-3t})^2$$

$$L = \int_0^{\ln(8)} \sqrt{(3e^{3t} + 3e^{-3t})^2}\, dt$$

$$= \int_0^{\ln(8)} (3e^{3t} + 3e^{-3t})\, dt$$

$$= \left(e^{3t} - e^{-3t}\right) \Big|_0^{\ln(8)}$$

$$= e^{3\ln(8)} - e^{-3\ln(8)}$$

$$= e^{\ln(8^3)} - e^{\ln\left(\frac{1}{8^3}\right)}$$

$$= \boxed{512 - \frac{1}{512}}$$

Example 3: Find the arclength of

$y = \sqrt{x}$ from $x=4$ to $x=9$.

First, parameterize by

$$x(t) = t, \quad y(t) = \sqrt{t} = t^{1/2}$$

$$x'(t) = 1, \quad y'(t) = \frac{1}{2\sqrt{t}}$$

Then

$$L = \int_4^9 \sqrt{1 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt$$

$$= \int_4^9 \sqrt{1 + \frac{1}{4t}}\, dt$$

$$L = \int_4^9 \sqrt{1 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt$$

$$= \int_4^9 \sqrt{1 + \frac{1}{4t}}\, dt$$

$$= \int_4^9 \sqrt{\frac{4t+1}{4t}}\, dt$$

$$= \int_4^9 \frac{1}{2\sqrt{t}} \sqrt{4t+1}\, dt$$

$u = \sqrt{t}$ $(u(4)=2,\ u(9)=3)$

$du = \frac{1}{2\sqrt{t}}\, dt$

$u^2 = t$ substitute!

$$L = \int_2^3 \sqrt{4u^2+1}\, du$$

$$u = \frac{1}{2}\tan\theta$$

$$du = \frac{1}{2}\sec^2\theta\, d\theta$$

forget bounds

$$\int \sqrt{4\left(\frac{1}{2}\tan\theta\right)^2+1}\ \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sqrt{\underbrace{\tan^2\theta+1}_{\sec^2\theta}}\ \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\, \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta$$

$\int \sec^3\theta \, d\theta$ is

perhaps the hardest integral

in this class!

Will work out later

Surface Area

(Section 8.2 - kind of)

Take a parametric curve

$f(t) = \langle x(t), y(t) \rangle$ with

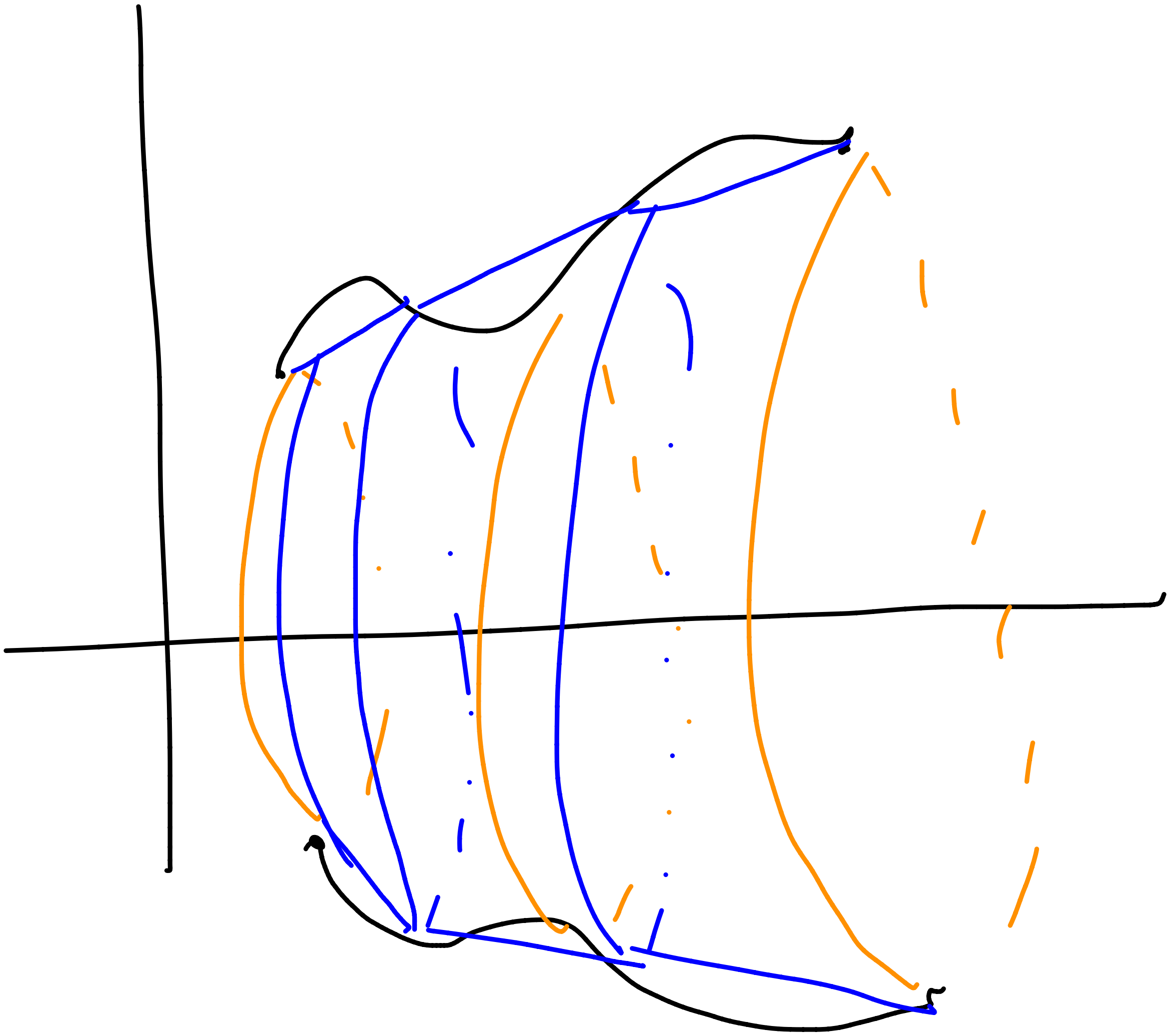
no self-intersections

Spin the curve around the

x or y axis from $t = a$

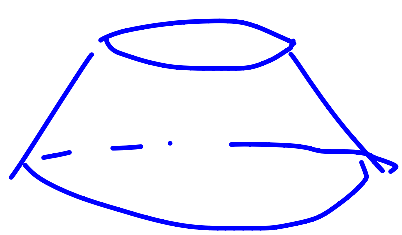
to $t = b$.

Picture



Want surface area, not volume!

Break up into pieces that look like



Take limits as number of subdivisions goes to infinity.

You get:

$$SA = \int_a^b \underbrace{2\pi|y(t)|}_{\text{circumference}} \underbrace{\sqrt{(x'(t))^2+(y'(t))^2}}_{\text{length of curve}}\,dt$$

(about x-axis)

or

$$SA = \int_a^b 2\pi|x(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

(about y-axis)