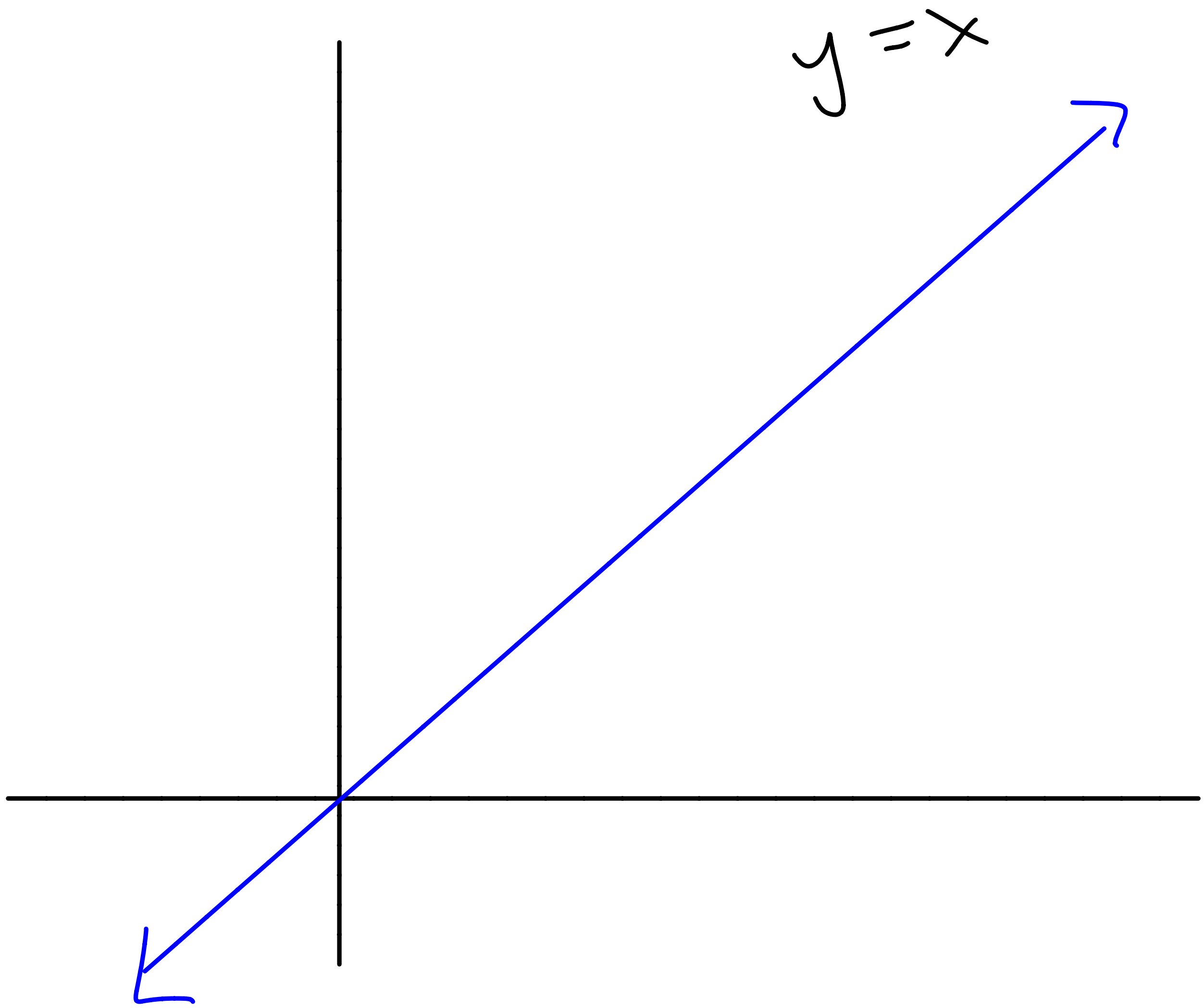
Curvature

This is in section 13.3 for curves in 3 dimensions, but we will do it for 2, where it is still interesting

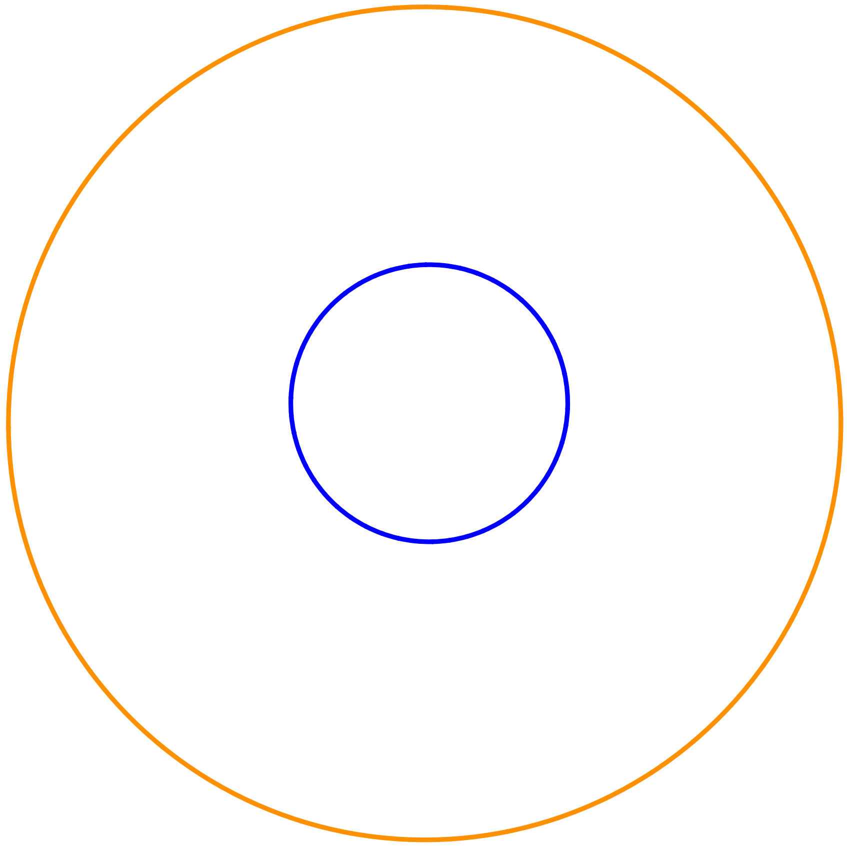
Example 1: any line

$$y = x$$



Curvature should be zero for this and any line!

Example 2: circles



Curvature is the same at any point on the circle (rotational symmetry), depends only on the radius of the circle - smaller radius = bigger curvature

Parametric Tangent

For a parametric curve

$f(t) = \langle x(t), y(t) \rangle$,

the tangent line at $t=a$ was

$$\frac{y'(a)}{x'(a)} = \frac{y - y(a)}{x - x(a)}$$

Can we define curvature to be the rate of change of the slope as we move around the curve? Doesn't work for vertical tangents

Back to arclength:

$$L(t) = \int_a^t \sqrt{(x'(s))^2 + (y'(s))^2}\, ds$$

$$\frac{dL}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

> 0

if x', y' are never simultaneously zero.

This means that the arclength is increasing, so L is invertible

The definition: If we translate the tangent line to the origin, we can measure the angle the line makes with the x-axis. Call this θ.

We can define the curvature to be

$$\boxed{\frac{d\theta}{dL}}$$ where L = arclength.

Makes the curvature independent of parameterization.

Curvature Formula

θ = angle

$\tan(\theta(t))$ = slope of line

$$= \frac{y'(t)}{x'(t)}$$

By chain rule,

$$\frac{d\theta}{dt} = \frac{d\theta}{dL}\frac{dL}{dt}$$

$$= \frac{d\theta}{dL}\sqrt{(x'(t))^2 + y'(t)^2}$$

So

$$\text{Curvature} = \frac{d\theta}{dL} = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}}\frac{d\theta}{dt}$$

Need: $\frac{d\theta}{dt}$

$$\text{Curvature} = \frac{d\theta}{dL} = \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}} \frac{d\theta}{dt}$$

Need: $\frac{d\theta}{dt}$

But $\tan(\theta(t)) = \frac{y'(t)}{x'(t)}$

so differentiate both sides.

$$\underbrace{\sec^2(\theta(t))\frac{d\theta}{dt}}_{\text{chain rule}} = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^2}$$

$$\sec^2(\theta(t)) = 1 + \tan^2(\theta(t))$$

$$= 1 + \left(\frac{y'(t)}{x'(t)}\right)^2$$

$$= \frac{x'(t)^2 + y'(t)^2}{(x'(t))^2}$$

So

$$\frac{d\theta}{dt} = \frac{1}{\sec^2(\theta(t))} \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^2}$$

$$= \frac{\cancel{(x'(t))^2}}{(x'(t))^2 + (y'(t))^2} \cdot \frac{x'(t)y''(t) - y'(t)x''(t)}{\cancel{(x'(t))^2}}$$

$$= \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^2 + (y'(t))^2}$$

$$\text{Curvature} = \frac{d\theta}{dL} = \frac{d\theta}{dt} \cdot \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

$$= \boxed{\frac{x'(t)y''(t) - y'(t)x''(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}}$$

Back to Lines and Circles

Lines: Let $y = mx + b$ be a line.

Parameterize as $\langle t, mt+b \rangle$

$$x'(t) = 1, \quad y'(t) = m$$

$$x''(t) = y''(t) = 0 \quad \text{so}$$

$$\text{curvature} = \frac{x'(t)\overbrace{y''(t)}^{=0} - y'(t)\overbrace{x''(t)}^{=0}}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}$$

$$= \frac{0}{(1+m^2)^{3/2}} = 0 \checkmark$$

If $x = a$ is a vertical line,

parameterize by

$$\langle a, t \rangle$$

$$x'(t) = 0 \quad , \; x''(t) = 0$$

$$y'(t) = 1 \; , \; y''(t) = 0$$

and we again get that the curvature is zero.

Circles: For simplicity, let's only do circles with center $(0,0)$

For $x^2+y^2=r^2$, a parameterization is $x(t)=r\cos(t)$, $y(t)=r\sin(t)$

$$x'(t)=-r\sin(t),\quad y'(t)=r\cos(t)$$

$$x''(t)=-r\cos(t)\quad y''(t)=-r\sin(t)$$

$$\text{Curvature}=\frac{|x'(t)y''(t)-y'(t)x''(t)|}{\left((x'(t))^2+(y'(t))^2\right)^{3/2}}$$

$$=\frac{-r\sin(t)(-r\sin(t))-r\cos(t)(-r\cos(t))}{\left(r^2\sin^2(t)+r^2\cos^2(t)\right)^{3/2}}$$

$$= \frac{-r\sin(t)(-r\sin(t)) - r\cos(t)(-r\cos(t))}{\left(r^2\sin^2(t) + r^2\cos^2(t)\right)^{3/2}}$$

$$= \frac{r^2\sin^2(t) + r^2\cos^2(t)}{\left(r^2\sin^2(t) + r^2\cos^2(t)\right)^{3/2}}$$

$$= \frac{r^2\left(\overbrace{\sin^2(t) + \cos^2(t)}^{=1}\right)}{\left(r^2\left(\underbrace{\sin^2(t) + \cos^2(t)}_{=1}\right)\right)^{3/2}}$$

$$= \frac{r^2}{(r^2)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}$$

Example 3: $y = x^2$

Parameterize as $\langle t, t^2 \rangle$

$x'(t) = 1, \quad x''(t) = 0$

$y'(t) = 2t, \quad y''(t) = 2$

$$\text{Curvature} = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}$$

$$= \frac{1 \cdot 2 - 2t \cdot 0}{\left(1 + (2t)^2\right)^{3/2}}$$

$$= \frac{2}{(1 + 4t^2)^{3/2}}$$

Depends on the value of t

For example, at $(0,0)$, $t=0$,

So the curvature is

$$\frac{2}{(1+0)^{3/2}} = 2,$$

but at $(1,1)$, $t=1$

and the curvature is

$$\frac{2}{(1+4)^{3/2}} = \frac{2}{5^{3/2}}$$

Nonconstant curvature.

What do you think the curvature go to as t increases?