Announcements

1) Exams back tomorrow

2) HW 3 up no later than Thursday, due next Thursday

3) Fall break Monday and Tuesday next week

4) Hint for extra credit: comparison!

Back to 2-Valve Problem

After applying the Laplace Transform to the differential equation

$$\frac{dx}{dt} = f(t) - \frac{3x(t)}{500}$$

where $f(t) = \begin{cases} 1.2, & t < 10 \\ 2.4, & t \geq 10 \end{cases},$

we got:

$$\boxed{\mathcal{L}(x)(s) = \frac{1500}{500s+3} + \frac{600e^{-10s}}{(500s+3)s} + \frac{600}{(500s+3)s}}$$

Strategy: For each term in the sum, find a function whose Laplace Transform is that term. Then $x(t)$ will be the sum of these functions!

Facts

1) $\mathcal{L}(e^{-at})(s) = \dfrac{1}{a+s}$, $s > -a$

2) $\mathcal{L}(f(t-a)h(t-a))(s) = e^{-sa}\mathcal{L}(f)(s)$

$(s > a)$

where

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\left(\mathcal{L}(f(t-a)h(t-a))(s) = \int_a^\infty f(t-a)e^{-st}\,dt \right)$$

$$\frac{1500}{500s+3} = 1500\left(\frac{1}{500s+3}\right)$$

$$= \frac{1500}{500}\left(\frac{1}{s+\frac{3}{500}}\right)$$

$$= 3\,\mathcal{L}\left(e^{-\frac{3t}{500}}\right)$$ ✓ using fact 1)

First term complete

$$\frac{600}{(500s+3)s} = 600\,\frac{1}{(500s+3)s}$$

$$= \frac{600}{500}\,\frac{1}{\left(s+\frac{3}{500}\right)s}$$

$$= \frac{6}{5}\,\frac{1}{\left(s+\frac{3}{500}\right)s}$$

Need a trick to be able to quote from our list of known Laplace Transforms. The trick is . . .

Partial Fractions

(section 7.4)

A way of decomposing a rational function $\frac{p(x)}{q(x)}$ where p and q are polynomials into a polynomial plus terms of the form $\frac{C}{x+a}$ or $\frac{D}{x^2+b}$

Example 1: $\dfrac{1}{s(500s+3)}$

I wish that there were numbers A and B with

$$\boxed{\frac{1}{s(500s+3)} = \frac{A}{s} + \frac{B}{500s+3}}$$

Solve for A and B!

Multiply both sides by $s(500s+3)$ to get:

$$1 = \frac{A}{\cancel{s}}\left(\cancel{s}(500s+3)\right) + \frac{B}{\cancel{(500s+3)}}\left(s\cancel{(500s+3)}\right)$$

$$= A(500s+3) + Bs$$

$$1 = A(500s+3) + Bs$$

Let $s = 0$

Then $1 = 3A$, so $A = \frac{1}{3}$

Let $s = \frac{-3}{500}$

Then $1 = B\left(\frac{-3}{500}\right)$, so $B = \frac{-500}{3}$.

This shows

$$\boxed{\frac{1}{s(500s+3)} = \frac{1}{3}\left(\frac{1}{s}\right) - \frac{500}{3}\left(\frac{1}{500s+3}\right)}$$

Finishing the Two Value Problem

$$\frac{600}{(500s+3)s} = 600 \frac{1}{(500s+3)s}$$

$$= 600\left(\frac{1}{3s} - \frac{500}{3}\frac{1}{500s+3}\right)$$

(previous page)

$$= 600\left(\frac{1}{3s} - \frac{1}{3}\cdot\frac{1}{s+\frac{3}{500}}\right)$$

$$= 200\left(\frac{1}{s} - \frac{1}{s+\frac{3}{500}}\right)$$

$$= 200\left(\mathcal{L}(1)(s) - \mathcal{L}\left(e^{-\frac{3t}{500}}\right)(s)\right)$$

$$= 200\left(\mathcal{L}\left(1-e^{-\frac{3t}{500}}\right)(s)\right)$$

Third term

Second term

$$\frac{600e^{-10s}}{(500s+3)s} = e^{-10s}(\text{third term})$$

$$= e^{-10s}\mathcal{L}\left(200-200e^{-\frac{3t}{500}}\right)(s)$$

$$= \mathcal{L}\left(h(t-10)\cdot 200\cdot\left(1-e^{-\frac{3(t-10)}{500}}\right)\right)(s)$$

using Fact 2)

✓

Finally!

$$x(t) = \underbrace{3e^{-\frac{3t}{500}}}_{1^{st}\text{ term}} + \underbrace{200h(t-10)\left(1-e^{-\frac{3(t-10)}{500}}\right)}_{2^{nd}\text{ term}}$$

$$+ \underbrace{200\left(1-e^{-\frac{3t}{500}}\right)}_{3^{rd}\text{ term}}$$

Other Situations (partial fractions)

1) Repeated linear factors

$$\frac{1}{(x-2)(x-3)^2} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

again cross-multiply

$$1 = A(x-3)^2 + B(x-2)(x-3) + C(x-2)$$

Choose: $x=2$, $x=3$, maybe $x=0$ to find B

2) Irreducible Quadratics

Quadratics like

x^2+1 that don't factor

into linear terms with

real coefficients.

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Again cross-multiply

$$1 = A(x^2+1) + (Bx+C)x$$

$$= Ax^2 + A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + A$$

$$1 = (A+B)x^2 + Cx + A$$

If these two polynomials are equal, then

$$A = 1$$

$$C = 0$$

$$A + B = 0, \text{ so } B = -1$$

3) Repeated Irreducible Quadratic Factors

$$\frac{1}{(x+2)(x-3)^2(x^2+4)^2}$$

$$= \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4}$$

$$+ \frac{Fx+H}{(x^2+4)^2}$$

Clear denominators and isolate coefficients like last example

$$1 = A(x-3)^2(x^2+4)^2$$

$$+B(x+2)(x-3)(x^2+4)^2$$

$$+C(x+2)(x^2+4)^2$$

$$+(Dx+E)(x+2)(x-3)^2(x^2+4)$$

$$+(Fx+H)(x+2)(x-3)^2$$

Expand using Wolfram Alpha!

$$1 = A(x^6 - 6x^5 + 17x^4 - 48x^3 + 88x^2 - 96x + 144)$$

$$+B(x^6 - x^5 + 2x^4 - 8x^3 - 32x^2 - 16x - 96)$$

$$+C(x^5 + 2x^4 + 8x^3 + 16x^2 + 16x + 32)$$

$$+D(x^6 - 4x^5 + x^4 + 2x^3 - 12x^2 + 72x)$$

$$+E(x^5 - 4x^4 + x^3 + 2x^2 - 12x + 72)$$

$$+F(x^4 - 4x^3 - 3x^2 + 18x)$$

$$+H(x^3 - 4x^2 - 3x + 18)$$

Collect like powers

$$1 = (A+B+D)x^6 + (-6A-B+C-4D+E)x^5$$
$$+(17A+2B+2C+D-4E+F)x^4$$
$$+(-48A-8B+8C+2D+E-4F+H)x^3$$
$$+(88A-32B+16C-12D+2E-3F-4H)x^2$$
$$+(-96A-16B+16C+72D-12E+18F-3H)x$$
$$+(144A-96B+32C+72E+18H)$$

Equate polynomials

$$1 = 144A - 96B + 32C + 72E + 18H$$

$$\begin{aligned}
0 &= A + B + D \\
&= -6A - B + C - 4D + E \\
&= 17A + 2B + 2C + D - 4E + F \\
&= -48A - 8B + 8C + 2D + E - 4F + H \\
&= 88A - 32B + 16C - 12D + 2E - 3F - 4H \\
&= -96A - 16B + 16C + 72D - 12E + 18F - 3H \\
&= 144A - 96B + 32C + 72E + 18H
\end{aligned}$$

Solve for Coefficients

Using either painful algebra or Wolfram Alpha

$$A = \frac{1}{1600}$$

$$B = \frac{-73}{54925}$$

$$C = \frac{1}{845}$$

$$D = \frac{99}{140608}$$

$$E = \frac{285}{70304}$$

$$F = \frac{7}{1352}$$

$$H = \frac{17}{676}$$

Inverse Trig & Trig Substitution

(sections 6.6 and 7.3)

A way to solve certain integrals that were inaccessible before. Like:

$$\int_0^{10} \sqrt{100-x^2}\, dx = \frac{100\pi}{4} \quad \text{(geometry)}$$

$$\int_0^{9} \sqrt{100-x^2}\, dx = ?$$

Example 1: $\displaystyle\int \frac{1}{1+x^2}\,dx$

If $x = \tan\theta$, then

$dx = \sec^2\theta\,d\theta$, and

the integral becomes

$$\int \frac{1}{\underbrace{1+\tan^2\theta}_{\sec^2\theta}} \sec^2\theta\,d\theta \quad \text{(trig identity)}$$

$$= \int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \int 1\,d\theta = \theta + C$$

If $x = \tan\theta$, then if

arctan is the inverse function to tangent, apply arctan to both sides:

$$\arctan(x) = \arctan(\tan\theta)$$

$$= \theta$$

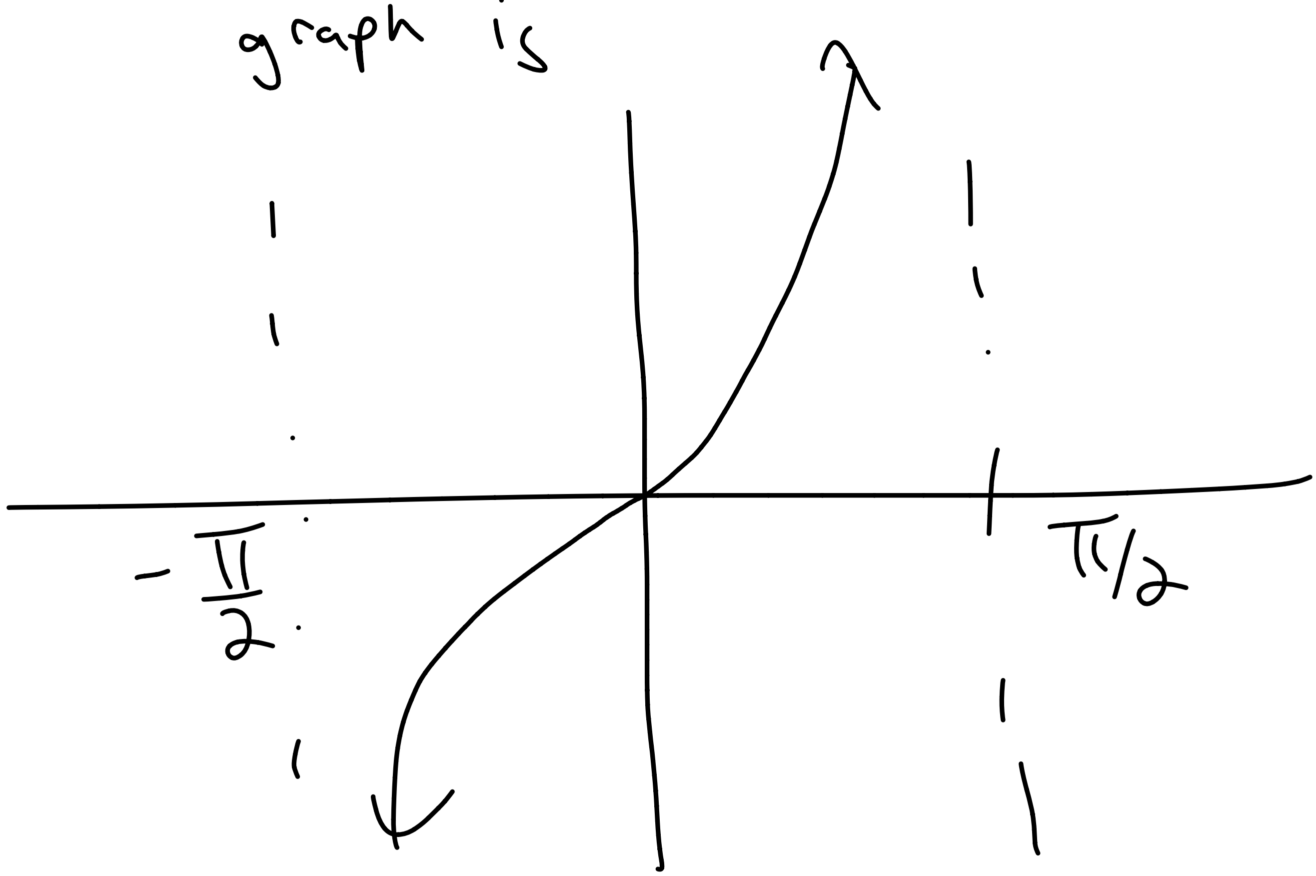
Final answer: $\boxed{\arctan(x) + C}$

Note: we showed

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

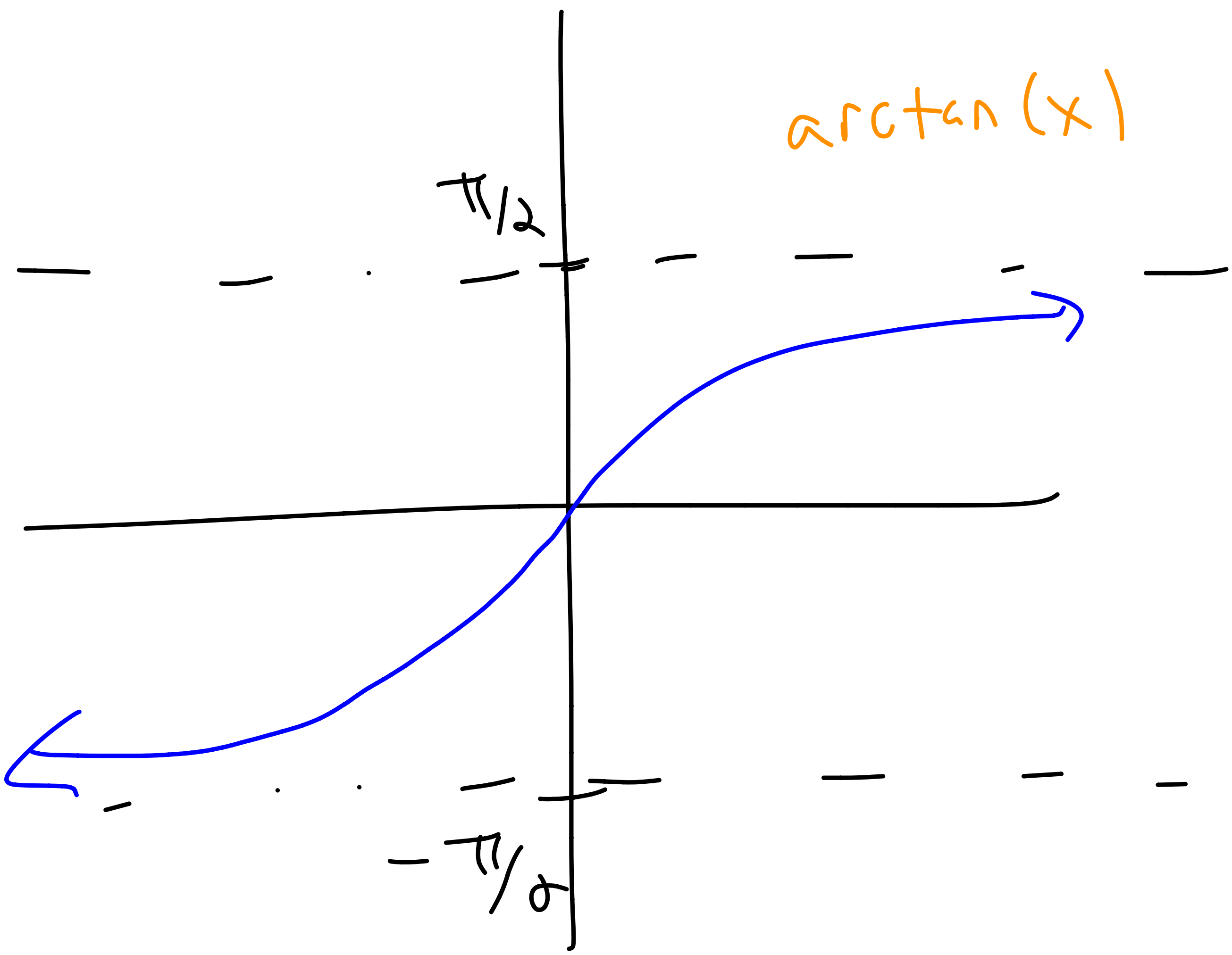
Restriction of Domain

Tangent is not invertible on its domain. Graph of tangent fails horizontal line test! But, when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, tangent's graph is



The "Inverse" of Tangent

Defined for all real numbers, range is from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and the graph is:



The Other (relevant) Inverse Trig Functions

1) arcsine: defined on $[-1, 1]$,

range is $[-\pi/2, \pi/2]$

$$\boxed{\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}}$$

2) arcsecant: defined on $(-\infty, 1]$ and

$[1, \infty)$, range is $[0, \pi/2)$ and $(\pi/2, \pi]$

$$\boxed{\frac{d}{dx}(\operatorname{arcsec}(x)) = \frac{1}{x\sqrt{x^2-1}}}$$