Announcements

1) Exam 1 Thursday

2) Extra Office Hours
   Wednesday 12-2

Extending the Natural Log

$$\int_{-1}^{-x} \frac{1}{t}\,dt, \quad \boxed{x > 0}$$

Let $u = -t$ $\quad u(-1) = 1$

$du = -dt$ $\quad u(-x) = x$

The integral becomes

$$\int_{1}^{x} \frac{1}{-u}(-du) = \int_{1}^{x} \frac{1}{u}\,du$$

$$= \ln(u)\Big|_{1}^{x}$$

$$= \ln(x) - \ln(1)$$

$$= \ln(x)$$

So in general, for $x \neq 0$,

$$\boxed{\int \frac{1}{x} = \ln(|x|) + C}$$

Example 1 :

$$\lim_{t \to \infty} \left(1+\frac{1}{t}\right)^t \neq 1$$

$$\lim_{t \to \infty} \left(1+\frac{1}{t}\right) = 1$$

$$\lim_{t \to \infty} t = \infty ,$$

1^{∞} is an indeterminate form

Need more work!

trick: $\left(1+\frac{1}{t}\right)^t = e^{\ln\left(\left(1+\frac{1}{t}\right)^t\right)}$

$= e^{t\ln\left(1+\frac{1}{t}\right)}$

If we want $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t$, we need only calculate $\lim_{t\to\infty} t\ln\left(1+\frac{1}{t}\right)$.

Then exponentiate to get the value of the limit.

$$\lim_{t\to\infty} t\ln(1+\tfrac{1}{t})$$

$$\lim_{t\to\infty} t=\infty$$

$$\lim_{t\to\infty} \ln(1+\tfrac{1}{t})=\ln(1)=0$$

$0\cdot\infty$ another indeterminate form!

Need a fraction to use l'Hopital's Rule.

$$t\ln(1+\tfrac{1}{t}) = \frac{\ln(1+\frac{1}{t})}{(1/t)}$$

$$= \frac{\ln(1+t^{-1})}{t^{-1}} = \underset{\text{limit}}{\frac{0}{0}}$$

Use l'Hopital's Rule

$$\lim_{t\to\infty} \frac{\ln(1+t^{-1})}{t^{-1}}$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{\frac{1}{1+t^{-1}} \frac{d}{dt}(1+t^{-1})}{-t^{-2}} \quad \left(= \frac{d}{dt} t^{-1}\right)$$

Chain rule

$$= \lim_{t\to\infty} \frac{\frac{1}{1+t^{-1}} \cancel{(-t^{-2})}}{\cancel{-t^{-2}}}$$

$$= \lim_{t\to\infty} \frac{1}{1+t^{-1}} = 1$$

Final answer: $e^1 = \boxed{e}$

Catalog of Indeterminate Forms

(L'Hopital's Rule)

1) $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

2) $0 \cdot (\pm\infty)$ (make a fraction)

3) 1^{∞} ($e^{\ln}$ trick)

4) 0^{0} ($e^{\ln}$ trick)

5) ∞^{0} ($e^{\ln}$ trick)

6) $\infty - \infty$ (common denominator or $e^{\ln}$ trick)

Example 2: $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$

$$= \int_{-2}^{1} x^{-2}\,dx$$

Note: integrand is undefined at $x=0$, so we can't use Fundamental Theorem! This is a special kind of improper integral.

More Improper Integrals

(Section 7.8)

Discontinuity in Interval

Let f be continuous on $[a,b]$ except at a point $x=c$ with $a<c<b$. Then we define

$$\boxed{\begin{aligned}\int_a^b f(t)\,dt &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt \\ &= \lim_{x\to c^-}\int_a^x f(t)\,dt \\ &\quad + \lim_{x\to c^+}\int_x^b f(t)\,dt\end{aligned}}$$

Back to Example

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{1} x^{-2}\,dx$$

$$= \int_{-2}^{0} x^{-2}\,dx + \int_{0}^{1} x^{-2}\,dx$$

$$= \lim_{s \to 0^-} \int_{-2}^{s} x^{-2}\,dx$$

$$+ \lim_{q \to 0^+} \int_{q}^{1} x^{-2}\,dx$$

Do the second integral:

$$\int_a^1 x^{-2}\,dx = -x^{-1}\Big|_a^1$$

$$= -1 + \frac{1}{a}$$

Take limit as $a \to 0^+$

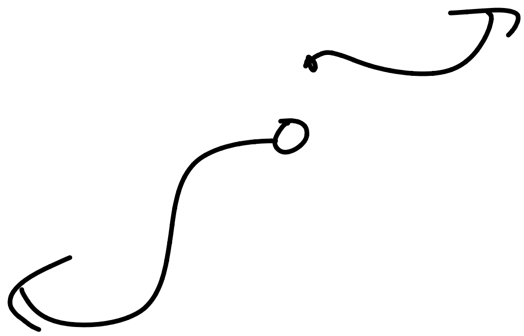
$$\lim_{a \to 0^+} \left(-1 + \frac{1}{a}\right) = \infty$$

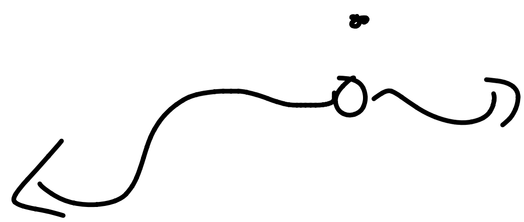
So the entire integral

does not exist

Jump and Removable Discontinuities

We don't have to take a limit:

Jump = 

Removable = 

If f has either of these discontinuities at $t = c$, $a < c < b$,

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

Back to the Two-Valve Problem

$$\frac{dx}{dt} = \begin{cases} f_1 \ell_1 - f_1\left(\frac{x(t)}{v}\right), & t < t_0 \\ f_1 \ell_2 - f_1\left(\frac{x(t)}{v}\right), & t \geq t_0 \end{cases}$$

Let $t_0 = 10$ min, $f_1 = 6$ L/min

$\ell_1 = .2$ kg/L , $\ell_2 = .4$ kg/L

$x(0) = 30$ kg , $v = 1000$ L

$$\frac{dx}{dt} = \begin{cases} 1.2 - \frac{6x(t)}{1000}, & t < 10 \\ 2.4 - \frac{6x(t)}{1000}, & t \geq 10 \end{cases}$$

$$\frac{dx}{dt} = \begin{cases} 1.2 - \frac{6x(t)}{1000}, & t < 10 \\ 2.4 - \frac{6x(t)}{1000}, & t \geq 10 \end{cases}$$

Note $\frac{6}{1000} = \frac{3}{500}$.

Let $f(t) = \begin{cases} 1.2, & t < 10 \\ 2.4, & t \geq 10 \end{cases}$

Our differential equation becomes

$$\boxed{\frac{dx}{dt} = f(t) - \frac{3x(t)}{500}}$$

$$\frac{dx}{dt} = f(t) - \frac{3x(t)}{500}$$

Take Laplace Transform of both sides!

$$\mathcal{L}\left(\frac{dx}{dt}\right)(s) = s\mathcal{L}(x)(s) - x(0)$$

The Laplace Transform is an integral, so it distributes over subtraction and pulls out constants when multiplied, so

$$\mathcal{L}\left(f(t) - \frac{3x(t)}{500}\right)(s)$$

$$= \mathcal{L}(f)(s) - \frac{3}{500}\mathcal{L}(x)(s)$$

Solving for $\mathcal{L}(x)(s)$,

$$\mathcal{L}(x)(s)\left(s+\frac{3}{500}\right)$$

$$= x(0) + \mathcal{L}(f)(s)$$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{x\to\infty} \int_0^x e^{-st} f(t)\,dt$$

$$f(t) = \begin{cases} 1.2, & t<10 \\ 2.4, & t\geq 10 \end{cases}$$, so we get

$$= \int_0^{10} 1.2e^{-st}\,dt + \lim_{x\to\infty} \int_{10}^x 2.4e^{-st}\,dt$$

$$= \left.\frac{1.2e^{-st}}{-s}\right|_0^{10} + \lim_{x\to\infty} \left.\frac{2.4e^{-st}}{-s}\right|_{10}^{x}$$

$$= \frac{1.2e^{-10s}}{-s} + \frac{1.2}{s} + \lim_{x\to\infty} \underbrace{\frac{-2.4}{e^{-sx}}}_{=0 \text{ if } s>0} + \frac{2.4e^{-10s}}{s}$$

$$= \frac{1.2(e^{-10s}+1)}{s}$$

Putting this all together,

$$\mathcal{L}(x)(s)\left(s+\frac{3}{500}\right) = x(0) + \frac{1.2(e^{-10s}+1)}{s}$$

$$= 30 + \frac{1.2(e^{-10s}+1)}{s}$$

Solving for $\mathcal{L}(x)(s)$,

$$\mathcal{L}(x)(s)\left(\frac{500s+3}{500}\right) = 30 + \frac{1.2(e^{-10s}+1)}{s}$$

$$\mathcal{L}(x)(s) = \frac{1500}{500s+3} + \frac{600(e^{-10s}+1)}{(500s+3)s}$$

$$\boxed{\mathcal{L}(x)(s) = \frac{1500}{500s+3} + \frac{600e^{-10s}}{(500s+3)s} + \frac{600}{(500s+3)s}}$$