Announcements

1) Quiz 1 tomorrow
over Calc 1 material

2) Review problems on
Canvas under "Files"

What is calculus good for?

Solving differential equations

What is a differential equation?

An equation involving an unknown function and its derivative(s)

Example 1:

$$\frac{dy}{dx} = x^2 + 1$$

y = unknown function

To get y, integrate both sides with respect to x:

$$\int \frac{dy}{dx}\, dx = \int (x^2+1)\, dx$$

$$y + C = \frac{x^3}{3} + x + D$$

$$y = \frac{x^3}{3} + x + (D - C)$$

Let $D - C = Q$

$$\boxed{y = \frac{x^3}{3} + x + Q}$$

Moral: you only need one arbitrary constant (in this class)

Example 2: (Newton's Law of Cooling)

Change in temperature in an object proportional to the difference between the temperature of an object and the ambient temperature.

Simplifying assumption: ambient temperature T is constant

Change in temperature = instantaneous rate of change

Let $x(t)$ denote the temperature of the object at time t.

"proportional" means "a constant multiple of"

Let $k =$ the constant of proportionality

Newton's Law of Cooling

$$\frac{dx}{dt} = k(x(t) - T)$$

Can't directly integrate both sides! Because we don't know $x(t)$, so we can't calculate

$$\int x(t)\, dt$$

Try to solve: divide by

$x(t) - T$ on both sides:

$$\frac{1}{x(t)-T}\frac{dx}{dt} = k.$$ Try integrating.

$$\int \frac{1}{x(t)-T}\frac{dx}{dt}\,dt = \boxed{\begin{aligned}&\int k\,dt\\ &= kt + C\end{aligned}}$$

Let $u = x(t) - T$, $du = \frac{dx}{dt}\,dt$.

Integral becomes

$$\int \frac{1}{u}\,du$$

What is $\int \frac{1}{u}\,du$?

$u \neq 0$

Concentrate on $u \geq 1$.

Form the function

$$\boxed{\int_1^x \frac{1}{u}\,du = \ln(x)}$$

Log property:

$$\ln(a^b) = b\ln(a) \quad (a \geq 1, b > 0)$$

Verify this property for

$$\int_1^x \frac{1}{u}\,du \; : \quad \text{show}$$

$$\int_1^{a^b} \frac{1}{u}\,du = b\int_1^a \frac{1}{u}\,du$$

Let b = constant, a = variable.

Differentiate both sides with respect to a.

By the Fundamental Theorem of Calculus, the right-hand side is

$$\frac{d}{da}\left(b \int_1^a \frac{1}{u}\, du \right)$$

$$= b \left(\frac{d}{da} \left(\int_1^a \frac{1}{u}\, du \right) \right)$$

$$= b \cdot \frac{1}{a} = \frac{b}{a}$$

Again using Fundamental Theorem,

$$\frac{d}{da}\left(\int_1^{a^b} \frac{1}{u}\,du\right)$$

$$= \frac{1}{a^b} \cdot \frac{d}{da}\left(a^b\right)$$

$$= \frac{1}{a^b} \cdot b \cdot a^{b-1}$$

$$= \frac{1}{\cancel{a^b}} \cdot b \cdot \cancel{a^b} \cdot \frac{1}{a}$$

$$= \frac{b}{a}$$ ✓

This shows that two functions have the same derivative, so they differ by a constant:

$$\int_1^{a^b} \frac{1}{u}\,du - b\int_1^{a} \frac{1}{u}\,du = C$$

$C =$ constant for <u>all</u> values of a & b.

Choose $a=1$:

$$\underbrace{\int_1^{1} \frac{1}{u}\,du}_{0} - b\underbrace{\int_1^{1} \frac{1}{u}\,du}_{0} = C$$

So $C=0$

So we know $\ln(x)$ exists for

$x \geq 1$, $\dfrac{d(\ln(x))}{dx} = \dfrac{1}{x}$

For $0 < x \leq 1$

$$\ln(x) = -\int_x^1 \frac{1}{u}\,du$$

$$= \int_1^x \frac{1}{u}\,du$$

= same as before

There is a number e with $\ln(e)=1$. Then using log rule $\ln(a^b)=b\ln(a)$,

$$\ln(e^x)=x\ln(e)=x$$

So $f(x)=e^x$ is the inverse function to $\ln(x)$

Rules: For all real numbers a & b,

$$e^{a+b}=e^a\cdot e^b \qquad e^0=1$$

$$e^{ab}=(e^a)^b$$

Derivative: $x = \ln(e^x)$

Differentiate both sides wrt x and use the chain rule:

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\ln(e^x)\right)$$

$$1 = \frac{1}{e^x} \cdot \frac{d}{dx}(e^x)$$

multiply both sides by e^x:

$$\boxed{e^x = \frac{d}{dx}(e^x)}$$

Example 3:

$$\int \tan(x)\,dx$$

$\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

$$= \int \frac{\sin(x)}{\cos(x)}\,dx$$

$$u = \cos(x)$$

$$du = -\sin(x)\,dx$$

We get

$$-\int \frac{1}{u}\,du = -\ln(u) + C$$

$$= \boxed{-\ln(\cos(x)) + C}$$

Example 4:

$$\int_0^2 \frac{x^2\left(10^{\sqrt{x^3+1}}\right)}{\sqrt{x^3+1}}\,dx$$

Let $u = \sqrt{x^3+1} = (x^3+1)^{1/2}$

$$du = \tfrac{1}{2}\cdot 3x^2\cdot (x^3+1)^{-1/2}\,dx$$

$$= \tfrac{3}{2}\,\frac{x^2}{\sqrt{x^3+1}}\,dx$$

$$\tfrac{2}{3}\,du = \frac{x^2}{\sqrt{x^3+1}}\,dx$$

$$u(0) = \sqrt{0^3+1} = 1$$

$$u(2) = \sqrt{2^3+1} = \sqrt{9} = 3$$

The integral becomes

$$\frac{2}{3}\int_1^3 10^u \, du$$

What is $\int 10^u du$?

Write $10^u = e^{\ln(10^u)}$

$$= e^{u\ln(10)}$$

Substitute one more time!

Let $w = u\ln(10)$

$dw = \ln(10)\, du$, so

$$\frac{1}{\ln(10)} dw = du$$

$$w(1) = \ln(10)$$

$$w(3) = 3\ln(10) = \ln(1000)$$

We get the integral

$$\frac{2}{3}\frac{1}{\ln(10)}\int_{\ln(10)}^{\ln(1000)} e^{w}\,dw$$

$$= \frac{2}{3\ln(10)} e^{w}\Big|_{\ln(10)}^{\ln(1000)}$$

$$= \frac{2}{\ln(1000)}(1000-10)$$

$$= \frac{2\cdot 990}{\ln(1000)} = \boxed{\frac{1980}{\ln(1000)}}$$

Example 5: (the only hard derivative)

$$f(x) = x^x$$

Write $x^x = e^{\ln(x^x)}$

$$= e^{x\ln(x)}$$

Differentiate:

$$\frac{d}{dx}\left(x^x\right) = \frac{d}{dx}\left(e^{x\ln(x)}\right)$$

$$= e^{x\ln(x)}\frac{d}{dx}\left(x\ln(x)\right)$$

$$= x^x\left(x\cdot\frac{1}{x} + \ln(x)\right)$$

$$= \boxed{x^x\left(1+\ln(x)\right)}$$