Announcements

1) HW 4 due Thursday

2) Quiz next Thursday over 10.1, 10.2 - practice problems on Canvas under "Assignments"

3) Exam week after next (Thursday)

4) Wear costume on Halloween, get candy!

Parametric Curves

(Section 10.1)

A way to unify rectangular and polar graphs by specifying and x function and a y function, both dependent on a single variable t

Write: $\langle x(t), y(t) \rangle$

Example 1: $f(t) = \langle \cos(t), \sin(t) \rangle$

$x(t) = \cos(t)$ (x-coordinate)

$y(t) = \sin(t)$ (y-coordinate)

The graph is a circle, center $(0,0)$ and radius 1, since

$$x(t)^2 + y(t)^2$$

$$= \cos^2(t) + \sin^2(t)$$

$$= 1$$

How to graph:

The graph of a parametric function is only points $(x(t), y(t))$ — you suppress the t value when graphing, so we get a 2-dimensional graph: a "piece of string" in 2 dimensions

Example 2: $g(t) = \langle 2t+5, t-6 \rangle$

Find a Cartesian equation

$$x(t) = 2t+5$$
$$y(t) = t-6$$

Suppress t inputs

$$x = 2t+5$$
$$y = t-6$$

Solve for t in one equation. Try

$y = t-6$, so

$t = y+6$.

Plug into equation for x

$$x = 2t + 5$$
$$= 2(y+6) + 5$$
$$= 2y + 17$$

So solving for y,

$$\boxed{\frac{x-17}{2} = y, \quad \text{a line!}}$$

Example 3: Parameterize $y = x^2$.

There are infinitely many ways to do this, but perhaps the easiest is just to rename

$$x = t, \quad y = x^2 = t^2$$

so

$\boxed{\langle t, t^2 \rangle}$ is a parameterization.

Parameterizing $y=f(x)$

We can generalize this last example to any function $y=f(x)$.

A parameterization is

$$\boxed{\langle t, f(t) \rangle}$$

Example 4: Parameterize $\theta = \frac{\pi}{3}$

First, change to rectangular coordinates using

$$\theta = \arctan\left(\frac{y}{x}\right). \text{ So}$$

$$\frac{\pi}{3} = \text{artan}\left(\frac{y}{x}\right)$$

and taking tangent of both sides,

$$\underbrace{\tan\left(\frac{\pi}{3}\right)}_{= \sqrt{3}} = \tan\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{y}{x}$$

We have $\frac{y}{x} = \sqrt{3}$, so $y = \sqrt{3}x$

Parameterize by $\boxed{\langle t, \sqrt{3}t \rangle}$

Calculus with Parametric Curves

(Section 10.2)

Tangent Lines

Given a parametric curve

$$\langle x(t), y(t) \rangle$$

Assume y can be expressed as a function of x (at least on small intervals of t).

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$$

$$\text{"="} \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Slope of tangent line at $t=a$:

$$\frac{y'(a)}{x'(a)}$$

Shorthand equation for tangent line:

$$\frac{y'(a)}{x'(a)} = \frac{y-y(a)}{x-x(a)}$$

Example 1: Find the tangent line

to $f(t) = \langle t^t, \arcsin(t^2) \rangle$

at $t = \frac{1}{\sqrt{2}}$

Find $x'(t), y'(t)$.

$$y'(t) = 2t \frac{d}{dt}(\arcsin)(t^2)$$

$$= 2t \frac{1}{\sqrt{1-(t^2)^2}}$$

$$= \frac{2t}{\sqrt{1-t^4}}$$

$$x(t) = t^t = e^{\ln(t^t)}$$

$$= e^{t\ln(t)}$$

$$x'(t) = e^{t\ln(t)} \frac{d}{dt}\left(t\ln(t)\right)$$

$$= t^t \left(1 \cdot \ln(t) + t \cdot \frac{1}{t}\right)$$

$$= t^t\left(\ln(t) + 1\right)$$

Equation for line:

$$\frac{y'\left(\frac{1}{\sqrt{2}}\right)}{x'\left(\frac{1}{\sqrt{2}}\right)} = \frac{y - y\left(\frac{1}{\sqrt{2}}\right)}{x - x\left(\frac{1}{\sqrt{2}}\right)}$$

$$y\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\left(\frac{1}{\sqrt{2}}\right)^2\right)$$

$$= \arcsin\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{6}$$

$$x\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{\sqrt{2}}}$$

$$y'\left(\frac{1}{\sqrt{2}}\right) = \frac{2\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^4}}$$

$$= \frac{2}{\sqrt{2}\sqrt{1-\frac{1}{4}}}$$

$$= \frac{2}{\sqrt{2-\frac{2}{4}}} = \frac{2}{\sqrt{\frac{6}{4}}} = \frac{4}{\sqrt{6}}$$

$$x'\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\left(\ln\left(\frac{1}{\sqrt{2}}\right)+1\right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\left(-\frac{\ln(2)}{2}+1\right)$$

Final answer:

$$\boxed{\frac{\left(4/\sqrt{6}\right)}{\left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\left(1-\frac{\ln(2)}{2}\right)} = \frac{y-\frac{\pi}{6}}{x-\left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}}}$$

or

$$\frac{4/\sqrt{6}}{\left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\left(1-\frac{\ln(2)}{2}\right)}\left(x-\left(\frac{1}{\sqrt{2}}\right)^{1/\sqrt{2}}\right) = y-\frac{\pi}{6}$$