Announcements

1) Exams returned Thursday

2) HW on curvature due Thursday

3) GIS Day 9-4 Wed,
Kochoff Hall

4) HW 6 due Tuesday 11/21

Sequences

(Section 11.1)

Idea: a sequence is just a list of terms, which can be finite or infinite.

We will concentrate on infinite sequences of real numbers.

Precise Definition

A sequence of real numbers is a function f from the counting numbers to the real numbers.

Example 1: Let n be a counting number.

Let $f(n) = e^{-n}$.

$$f(1) = e^{-1}$$

$$f(2) = e^{-2}$$

$$f(3) = e^{-3}$$

$$\vdots$$

Shorthand

Instead of $f(n)$, we write

a_n or b_n or x_n etc.

For the previous example:

$$a_1 = e^{-1}$$

$$a_2 = e^{-2}$$

$$a_3 = e^{-3}$$

$$\vdots$$

Convergence

A sequence of real numbers $(a_n)_{n=1}^{\infty}$ is said to **converge** to a real number L if for all numbers $\varepsilon > 0$, there is a counting number N with

$$\boxed{|a_n - L| < \varepsilon \quad \text{when} \quad n \geq N.}$$

Example 1: $a_n = \frac{1}{n}$

$\lim_{n \to \infty} \frac{1}{n} = 0$, show this

using the definition.

Pick $\varepsilon > 0$. Show: there

is a number N with

$\left| \frac{1}{n} - 0 \right| < \varepsilon$ when

$n \geq N$.

Simplify: $\left| \frac{1}{n} - 0 \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$

Show $\frac{1}{n} < \varepsilon$ when $n \geq N$.

Show $\frac{1}{n} < \varepsilon$ when $n \geq N$.

Consider $\frac{1}{\varepsilon} > 0$. whatever this number is, I can find a counting number N with

$$N > \frac{1}{\varepsilon}.$$

Then $\varepsilon > \frac{1}{N}$, so for $n \geq N$

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon$$ done

Representation Trick

or: how to really compute sequential limits.

Given a sequence $(a_n)_{n=1}^{\infty}$ of real numbers, if there is a function f whose domain contains $[1,\infty)$ and

$\boxed{f(n) = a_n}$ for all counting numbers n, then if f is continuous,

$$\boxed{\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)}$$

Example 2: Find

$$\lim_{n\to\infty}\left(1+\frac{3}{e^n}\right)^{e^n}$$

Change n's to x's, evaluate

$$\lim_{x\to\infty}\left(1+\frac{3}{e^x}\right)^{e^x}$$

$e^{\ln}$ trick:

$$\left(1+\frac{3}{e^x}\right)^{e^x}=e^{\ln\left(\left(1+\frac{3}{e^x}\right)^{e^x}\right)}$$

$$=e^{e^x\ln\left(1+\frac{3}{e^x}\right)}$$

Figure out $\lim_{x\to\infty} e^x\ln\left(1+\frac{3}{e^x}\right)$, then exponentiate for the answer

Figure out $\lim_{x\to\infty} e^x \ln\left(1+\frac{3}{e^x}\right)$, then exponentiate for the answer

$$e^x \ln\left(1+\frac{3}{e^x}\right)$$

$$= \frac{\ln\left(1+\frac{3}{e^x}\right)}{\left(\frac{1}{e^x}\right)}$$

$$= \frac{\ln\left(1+3e^{-x}\right)}{e^{-x}}$$

$$\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \ln\left(1+3e^{-x}\right) = 0$$

So use l'Hopital's rule

$$\lim_{x\to\infty} \frac{\ln(1+3e^{-x})}{e^{-x}}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+3e^{-x}} \cdot 3\overbrace{(\cancel{-e^{-x}})}^{\text{chain rule}}}{\cancel{-e^{-x}}}$$

$$= \lim_{x\to\infty} \frac{3}{1+3e^{-x}} = 3$$

Final answer: $\boxed{e^3}$

The Squeeze Theorem

If $a_n \le b_n \le c_n$ are sequences of real numbers and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \text{, then}$$

$$\lim_{n\to\infty} b_n = L$$

Basic Limit Laws

If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of real numbers and $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, then

1) $\lim_{n\to\infty} (a_n + b_n) = L + M$

2) $\lim_{n\to\infty} (a_n - b_n) = L - M$

3) $\lim_{n\to\infty} (a_n b_n) = LM$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

5) If c is any constant,

$$\lim_{n \to \infty} c a_n = cL$$

6) If there is a counting number N with $a_n \leq b_n$ whenever $n \geq N$, then

$$L \leq M$$

7) If f is continuous,

$$\lim_{n \to \infty} f(a_n) = f(L)$$

Series

(Section 11.2)

Q: How do we "add up infinitely many numbers"?

A: Take a sequence $(a_n)_{n=1}^{\infty}$;

for example, $a_n = \frac{1}{2^n}$.

Define the partial sums of the sequence to be

$$\boxed{S_k = \sum_{n=1}^{k} a_n}$$

General $(a_n)_{n=1}^{\infty}$

$a_n = \frac{1}{2^n}$

$S_1 = a_1$

$S_1 = \frac{1}{2}$

$S_2 = a_1 + a_2$

$S_2 = \frac{1}{2} + \frac{1}{4}$

$S_3 = a_1 + a_2 + a_3$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$S_4 = a_1 + a_2 + a_3 + a_4$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$\vdots$

$S_k = \sum_{n=1}^{k} a_n$

$S_k = \sum_{n=1}^{k} \frac{1}{2^n}$

Note: $S_{k+1} = S_k + a_{k+1}$

We define the infinite series

$$\sum_{n=1}^{\infty} a_n \quad \text{to be}$$

$$\boxed{\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right)}$$

provided the limit exists!

- limit exists = convergent series
- limit does not exist = divergent series

Example 1:

$$a_n = \frac{1}{2^n}$$

$$S_{k+1} = \sum_{n=1}^{k+1} \frac{1}{2^n}$$

$$= \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}}_{S_k} + \frac{1}{2^{k+1}}$$

$$= S_k + \frac{1}{2^{k+1}}$$

But also,

$$S_{k+1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= \frac{1}{2} + \frac{1}{2}\Big(\underbrace{\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^k}}_{S_k}\Big)$$

$$= \frac{1}{2} + \frac{S_k}{2}$$

Equate the formulas for S_{k+1}:

$$S_k + \frac{1}{2^{k+1}} = S_{k+1} = \frac{1}{2} + \frac{S_k}{2}$$

So $S_k + \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{S_k}{2}$

$$- \frac{S_k}{2} - \frac{1}{2^{k+1}} \qquad - \frac{1}{2^{k+1}} - \frac{S_k}{2}$$

$$\frac{S_k}{2} = \frac{1}{2} - \frac{1}{2^{k+1}}$$

multiplying both sides by 2,

$$S_k = 1 - \frac{2}{2^{k+1}} = 1 - \frac{1}{2^k}$$

$$= 1 - \left(\frac{1}{2}\right)^k$$

$$S_k = 1 - \frac{2}{2^{k+1}} = 1 - \frac{2}{2^k \cdot 2^1}$$

$$= 1 - \frac{\cancel{2}}{2^k \cdot \cancel{2}}$$

$$= 1 - \frac{1}{2^k}$$

$$= 1 - \left(\frac{1}{2}\right)^k$$

Take $\lim\limits_{k \to \infty} S_k = \lim\limits_{k \to \infty} \left(1 - \left(\frac{1}{2}\right)^k\right)$

$$= 1 - \lim_{k \to \infty} \left(\frac{1}{2}\right)^k$$

$= 1 - 0$ since $0 < \frac{1}{2} < 1$

$$= \boxed{1}$$