Announcements

1) HW1 returning tomorrow, HW2 up on Canvas

Back to Example

$$\int_0^\infty t e^{-t}\, dt$$

$$= \lim_{x\to\infty} \int_0^x t e^{-t}\, dt$$

Substitute $u = -t$ $\quad u(0)=0$

$du = -dt$ $\quad u(x) = -x$

$$= \lim_{x\to\infty} \int_0^{-x} -u e^{u} (-du)$$

$$= \lim_{x\to\infty} \int_0^{-x} u e^{u}\, du$$

what now?

Integration by Parts

(Section 7.1)

Substitution = integrating the chain rule

Integration by parts = integrating the product rule

Definite and Indefinite Forms

$$\int f(t)g'(t)\,dt$$

$$= f(t)g(t) - \int g(t)f'(t)\,dt$$

$$\int_a^b f(t)g'(t)\,dt$$

$$= f(t)g(t)\Big|_a^b - \int_a^b g(t)f'(t)\,dt$$

Usually written in indefinite form as:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Given: $\int u\,dv$, transform into $uv - \int v\,du$

idea: choose u, dv in your integral

$dv =$ something you can integrate

$u =$ something you can differentiate

Returning to Example

$$\int_0^{-x} u e^u \, du$$

replace "u" with "s"

$$\int_0^{-x} s e^s \, ds$$

Use integration by parts

$u = s \qquad v = e^s$

$du = ds \qquad dv = e^s$

Then

$$\int_0^{-x} s e^{s}\,ds = s e^{s}\Big|_0^{-x} - \int_0^{-x} e^{s}\,ds$$

$$\left(\int u\,dv = uv - \int v\,du\right)$$

$$s e^{s}\Big|_0^{-x} = -x e^{-x}$$

$$\int_0^{-x} e^{s}\,ds = e^{s}\Big|_0^{-x} = e^{-x} - e^{0}$$
$$= e^{-x} - 1$$

Combining,

$$\int_0^{-x} s e^{s}\,ds = -x e^{-x} - \left(e^{-x} - 1\right)$$
$$= 1 - e^{-x} - x e^{-x}$$

Combining,

$$\int_0^{-x} s e^s \, ds = -xe^{-x} - (e^{-x} - 1)$$
$$= 1 - e^{-x} - xe^{-x}$$

$$= 1 - \frac{1}{e^x} - \frac{x}{e^x}$$

To solve the original problem, calculate

$$\lim_{x \to \infty} \int_0^{-x} s e^s \, dx = \lim_{x \to \infty} \left(1 - \frac{1}{e^x} - \frac{x}{e^x} \right)$$

$$\lim_{x \to \infty} 1 = 1 \qquad \lim_{x \to \infty} \frac{1}{e^x} = 0$$

$$\lim_{x \to \infty} \frac{x}{e^x} = \frac{\infty}{\infty} \quad ?$$

To calculate

$$\lim_{x \to \infty} \frac{x}{e^x}$$, we need

an additional trick, especially designed for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ quotients: L'Hopital's Rule!

L'Hopital's Rule

(Section 6.8)

Suppose $\lim_{x \to a} f(x) = 0$ and

$\lim_{x \to a} g(x) = 0$ (or $\lim_{x \to a} f(x) = \pm\infty$

and $\lim_{x \to \infty} g(x) = \pm\infty$). Suppose

f and g are differentiable in

an open interval containing $x = a$.

Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Comments:

1) $a = \pm\infty$ is also allowed, just need f and g differentiable on an appropriate interval

2) L'Hopital's Rule is **NOT** the quotient rule!

Finishing Example

$$\lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty},$$ use l'Hopital's Rule and indicate you are using it!

$$\lim_{x\to\infty} \frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{e^x}$$

$$= 0$$

So finally,

$$\int_0^\infty te^{-t}\,dt = \lim_{x\to\infty}\left(1 - \frac{1}{e^x} - \frac{x}{e^x}\right)$$

$$= 1 - 0 - 0 = \boxed{1}$$

More on Laplace Transforms

Recall: $$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Find $\mathcal{L}(f')(s)$ in terms of $\mathcal{L}(f)(s)$! (Provided f' exists)

$$\mathcal{L}(f')(s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{x\to\infty} \underbrace{\int_0^x e^{-st} f'(t)\,dt}_{\text{integrate by parts}}$$

$$\int_0^x e^{-st} f'(t)\,dt$$

$$u = e^{-st} \qquad v = f(t)$$

$$du = \underbrace{e^{-st}(-s)}\,dt \qquad dv = f'(t)\,dt$$

$$= -se^{-st}\,dt$$

Then

$$\int_0^x e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^x$$

$$+ \int_0^x s e^{-st} f(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^x + s\int_0^x e^{-st} f(t)\,dt$$

Note: $\lim\limits_{x\to\infty} s\int_0^x e^{-st} f(t)\,dt$

$$= s\lim_{x\to\infty}\int_0^x e^{-st} f(t)\,dt$$

$$= s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\,\mathcal{L}(f)(s)$$

$$\lim_{x\to\infty} e^{-st} f(t)\Big|_0^x$$

$$= \lim_{x\to\infty}\left(e^{-sx} f(x) - f(0)\right)$$

$$\lim_{x\to\infty} e^{-sx} f(x) = ?$$

I wish

$$\lim_{x \to \infty} e^{-sx} f(x) = 0.$$

In this case,

$$\boxed{\mathcal{L}(f')(s) = f(0) + s\mathcal{L}(f)(s)}$$

Laplace Transform of a Derivative

Example 1: $\displaystyle\int_0^{\pi/2} x^2\cos(x)\,dx$

integrate by parts:

$u = x^2 \qquad v = \sin(x)$

$du = 2x\,dx \qquad dv = \cos(x)\,dx$

So

$$\int_0^{\pi/2} x^2\cos(x)\,dx = x^2\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 2x\sin(x)\,dx$$

$$= \frac{\pi^2}{4} - \underbrace{\int_0^{\pi/2} 2x\sin(x)\,dx}_{\text{another integration by parts}}$$

$$\int_0^{\pi/2} 2x \sin(x)\, dx$$

$$u = 2x \qquad v = -\cos(x)$$

$$du = 2\, dx \qquad dv = \sin(x)\, dx$$

$$\int_0^{\pi/2} 2x \sin(x)\, dx = -2x\cos(x)\Big|_0^{\pi/2} + \int_0^{\pi/2} 2\cos(x)\, dx$$

$$= 2\int_0^{\pi/2} \cos(x)\, dx$$

$$= 2\sin(x)\Big|_0^{\pi/2}$$

$$= 2$$

Final answer: $\boxed{\frac{\pi^2}{4} - 2}$

Another Approach:

Tabular Trick!

For (polynomial) $\cdot$ $\left\{ \begin{array}{l} \text{exponentials} \\ \text{sines} \\ \text{cosines} \end{array} \right.$

1) Make a table

u	dv
x^2	$\cos(x)$

2) Differentiate u until you get zero

u	dv
x^2	$\cos(x)$
$2x$	
2	
0	

3) Integrate dv as many times as you differentiated u

u	dv
x^2	$\cos(x)$
$2x$	$\sin(x)$
2	$-\cos(x)$
0	$-\sin(x)$

4) Multiply diagonally with alternating signs, starting with "+"

u		dv
x^2	$+$	$\cos(x)$
$2x$	$-$	$\sin(x)$
2	$+$	$-\cos(x)$
0		$-\sin(x)$

5) Add all terms & plug in limits

$$\left(x^2\sin(x)+2x\cos(x)-2\sin(x)\right)\Big|_0^{\pi/2}$$

$$=\frac{\pi^2}{4}-2$$

Example 2: $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$

We know this limit is equal to one from Calc 1, but you get it easily from l'Hopital's Rule:

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

$$\overset{H}{=} \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = \boxed{1}$$

Example 3: $\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t$

$$\lim_{t\to\infty} \left(1+\frac{1}{t}\right) = 1$$

$$\lim_{t\to\infty} t = \infty$$

You can't conclude anything from this!

Rewrite as a quotient for L'Hopital's Rule using $e^{\ln(x)} = x$:

$$\left(1+\frac{1}{t}\right)^t = e^{\ln\left(\left(1+\frac{1}{t}\right)^{t}\right)}$$

$$= e^{t\ln\left(1+\frac{1}{t}\right)}$$

We only need the limit of the exponent

$$\lim_{t\to\infty} t\ln(1+\tfrac{1}{t})$$

$$\lim_{t\to\infty} t=\infty, \quad \lim_{t\to\infty}\ln(1+\tfrac{1}{t})=0$$

Rewrite as a quotient:

$$\lim_{t\to\infty}\frac{\ln(1+\frac{1}{t})}{1/t}$$

$$\overset{H}{=}\lim_{t\to\infty}\frac{\frac{1}{1+1/t}(1+1/t)'}{-1/t^2}$$

$$=\lim_{t\to\infty}\frac{\frac{1}{1+1/t}\cdot\cancel{(-1/t^2)}}{\cancel{-1/t^2}}$$

We get $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)=1$,

so the final answer is

$$e^1 = \boxed{e}$$