# Announcements

1) Office hours

M 11 – 12

T 3:30 – 4:30

Th 11 – 12

2) Quizzes – please give me back your quizzes! Gateway will be up later today – in Math Learning Center through Webwork!

3) Webwork – demo tomorrow.

HW1 due Thursday.

Do not use a typed-in URL for Webwork!

Access through "Modules" on Canvas.

# More Differential Equations

The dreaded brine problem!

A tank contains $V$ liters of a solution of water mixed with $m$ kilograms of a substance (usually salt).

Liquid contained $l_1$ Kg/L of the substance flows in at a rate of $f_1$ L/min and liquid exits the tank at the same rate.

Problem: Solve for $x(t)$, the amount of substance in the tank at time $t$ (in kg)

Derivative $\frac{dx}{dt}$, units are kg/min.

Setting up the problem

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \ell_1 f_1 - f_1\left(\frac{x(t)}{v}\right)$$

$$= f_1\left(\ell_1 - \frac{x(t)}{v}\right)$$

$$= f_1\left(\frac{v\ell_1 - x(t)}{v}\right)$$

Divide both sides by $(v\ell_1 - x(t))$ to get

$$\frac{1}{v\ell_1 - x(t)} \frac{dx}{dt} = \frac{f_1}{v}$$

$f_1, \ell_1, v$ are all constants

$$\frac{1}{v\ell_1 - x(t)} \frac{dx}{dt} = \frac{f_1}{v}$$

Integrate both sides with respect to t (independent variable)

$$\int \frac{1}{v\ell_1 - x(t)} \frac{dx}{dt}\, dt = \int \frac{f_1}{v}\, dt$$

$$= \frac{f_1}{v} t + C$$

$$u = v\ell_1 - x(t)$$

$$du = -\frac{dx}{dt}\, dt$$

$$-\int \frac{1}{u}\, du = \frac{f_1}{v} t + C$$

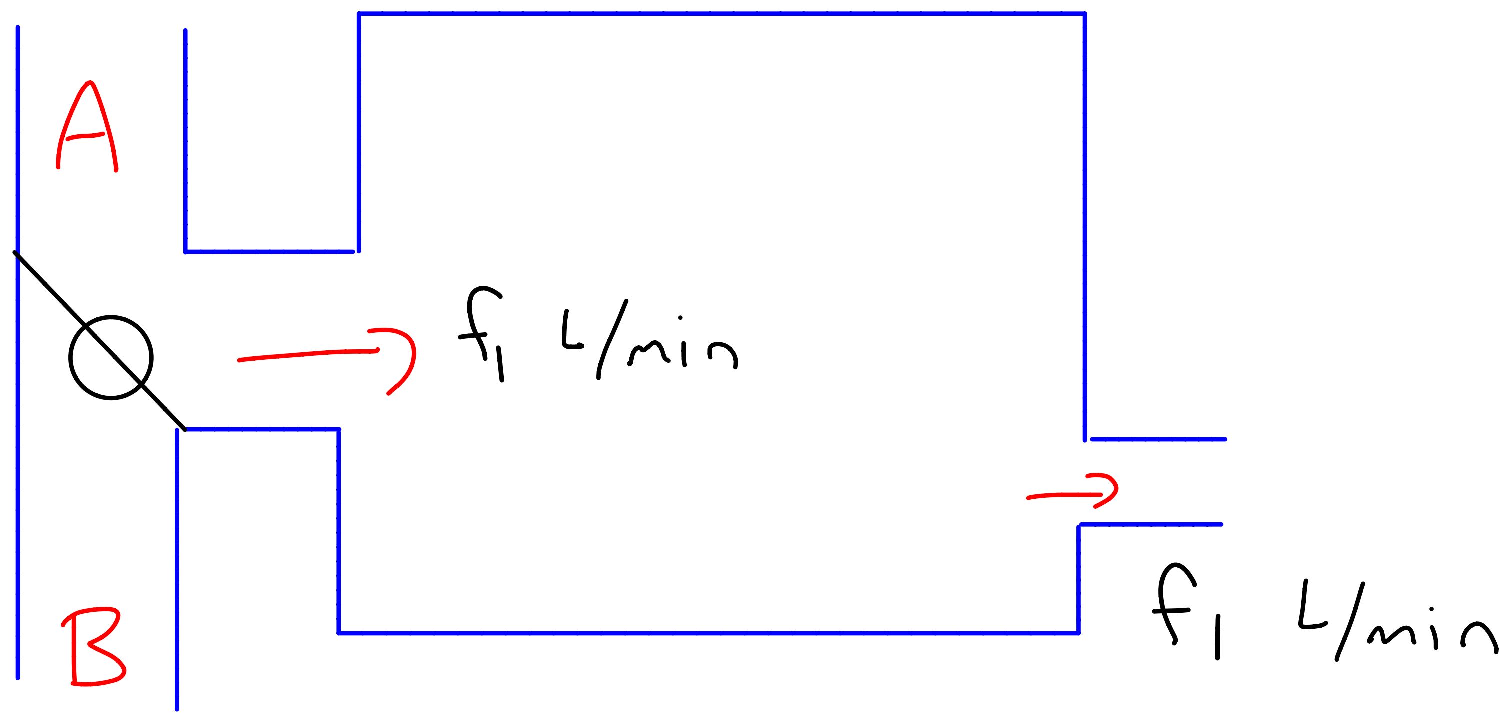
Integrating, we get

$$-\ln(v) = \frac{f_1}{r} t + C$$

# Ugly Variation

There are now two valves delivering liquid. The valve we had on (valve A) shuts off at time $t_0$, when valve B is turned on, delivering liquid containing $\ell_2$ kg/L of substance at $f_1$ L/min.

# Picture



## Setting up the problem

Solve for $x(t)$, the amount of substance in the tank at time $t$.

Same equation

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \begin{cases} (\text{original problem}), & t < t_0 \\ (\text{new rate in}), & t \geq t_0 \end{cases}$$

$$= \begin{cases} f_1 \ell_1 - f_1\left(\frac{x(t)}{v}\right), & t < t_0 \\ f_1 \ell_2 - f_1\left(\frac{x(t)}{v}\right), & t \geq t_0 \end{cases}$$

What we don't want to do:

Solve two problems like the original, one-valve problem.

What we do want to do:

Laplace transforms!

The Laplace transform of a function $f$ is given by

$$\boxed{\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st}\, dt}$$

$\mathcal{L}(f)$ is a function dependent on the variable "$s$".

What does that mean?

We have $\int_0^\infty f(t)e^{-st}\,dt$.

How do you define an integral over an infinite domain?

$$\int_0^\infty f(t)e^{-st}\,dt \text{ is defined as}$$

$$= \lim_{q\to\infty} \int_0^q f(t)e^{-st}\,dt$$

where $q>0$ is a real number

provided the limit exists!

If the limit exists, we say the integral converges to the value of the limit. If the limit does not exist, we say the integral diverges.

$$\int_0^\infty f(t)\,e^{-St}\,dt$$ is an example of an improper integral.

Example: $\int_0^\infty e^{-x}\,dx$

$$= \lim_{q\to\infty} \int_0^q e^{-x}\,dx$$

integrate

Substitute: $u = -x$

$du = -1\,dx$

so $-du = dx$

$u(0) = 0$

$u(q) = -q$

we get

$$\lim_{q\to\infty} - \int_0^{-q} e^{u}\,du$$

Integrating, we have

$$\lim_{q \to \infty} -\left( e^{u} \Big|_{0}^{-q} \right)$$

$$= \lim_{q \to \infty} -\left( e^{-q} - 1 \right)$$

$$= \lim_{q \to \infty} \left( 1 - e^{-q} \right)$$

$$= \lim_{q \to \infty} \left( 1 - \frac{1}{e^{q}} \right)$$

$$= 1 - \lim_{q \to \infty} \frac{1}{e^{q}}$$

$$= 1 - 0 = \boxed{1}$$

Limit results for logs and exponentials:

1) $\lim_{x \to \infty} e^x = \infty$

2) $\lim_{x \to -\infty} e^x = 0$

3) $\lim_{x \to \infty} \ln(x) = \infty$

4) $\lim_{x \to 0^+} \ln(x) = -\infty$

## Example 2:

$$\int_0^\infty x e^{-x}\,dx$$

$$= \lim_{q \to \infty} \int_0^q x e^{-x}\,dx$$

$u = -x$

$du = -1\,dx$, so $-du = dx$

We get $(u(0) = 0,\ u(q) = -q)$

$$\lim_{q \to \infty} \int_0^{-q} u e^{u}\,du$$

Need a new technique!