Name:

Math 216 Exam 3

December 3, 2015

Directions: WRITE YOUR NAME ON THIS EXAM! Except where indicated, merely finding the answer to a problem is not enough to receive full credit; you must show how you arrived at that answer.

1) For each function, compute the Laplace transform. Don't worry about the domain of definition.

a) (4 points) $\sin(5t) + t^4$.

b) (6 points) $2^t + \cos(3t)(\cos^2(t) + \sin^2(t))$

c) (10 points) $t\sin(3t) - e^{2t}\cos(t)$

2) A tank holds 250 L of brine initially containing 10 kg of dissolved salt. For the first 5 minutes of operation, valve A is open, adding 5L/min of brine containing .1kg/L of salt. After 5 minutes, valve A is switched off and valve B is switched on, delivering 5L/min of brine containing .2kg/L of salt. The exit valve C removes 5L/min of brine. Let $x(t)$ denote the amount of salt in the tank at time t.

a) (5 points) Find an equation for the rate of change of the concentration of salt coming INTO the tank.

b) (10 points) Determine an initial value problem that, when solved, will produce $x(t)$, but DO NOT SOLVE THE EQUATION.

3) (32 points) Solve the initial value problem

$$x'(t) = 5g(t) - \frac{x(t)}{50}, \quad x(0) = 10$$

where

$$g(t) = \begin{cases} .1, & 0 \le t \le 5 \\ .2, & t > 5. \end{cases}.$$

Note that you can write $g(t) = .1(u(t-5)+1)$ where u is the Heaviside function.

4) Suppose you are given a mass-spring oscillator initially at rest 3 m below equilibrium. At $t = 2\pi/3$ seconds, the mass is struck by a hammer with a force of 18 N. If the mass m is 3 kg, there are no external damping forces acting on the system, and the spring constant $k = 12$, find a formula for the position $y(t)$ of the mass by

a) (11 points) writing down a formal initial value problem describing this physical scenario,

b) (8 points) taking the Laplace transform of both sides of the equation you found in a),

c) (14 points) solving for the position $y(t)$ by taking the inverse Laplace transform of the equation you found in b).

BONUS: (10 points) Compute $\mathcal{L}(e^{it})(s)$.

LaPlace Transforms and Properties

1. $\mathcal{L}(e^{at})(s) = \dfrac{1}{s-a},\ s > a$

2. $\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}},\ s > 0$

3. $\mathcal{L}(\cos(at))(s) = \dfrac{s}{s^2+a^2},\ s > 0$

4. $\mathcal{L}(\sin(at))(s) = \dfrac{a}{s^2+a^2},\ s > 0$

5. $\mathcal{L}(\delta(t-a))(s) = e^{-as},\ s > 0$ where δ is the Dirac delta "function".

6. $\mathcal{L}(u(t-a)f(t-a))(s) = e^{-as}\mathcal{L}(f)(s)$ where u is the Heaviside function.

7. $\mathcal{L}(e^{at}f(t))(s) = \mathcal{L}(f)(s-a)$