Name:

Math 216 Final

December 18, 2015

Directions: WRITE YOUR NAME ON THIS EXAM! Except where indicated, merely finding the answer to a problem is not enough to receive full credit; you must show how you arrived at that answer. IF you convert irrational numbers such as $\sqrt{3}$ or π into decimal approximations, round to at least 4 decimal points.

Now, if you had a choice on whether to take this exam, you must also sign to indicate your understanding of the following statement:

I understand that in choosing to take this final, I may lower my grade from what it was before the final.

SIGNED:__________________________________

1) (3 points each) For each differential equation, determine its order and whether it is linear or homogeneous.

a) $x\dfrac{d^2y}{dx^2} - \sin(x)\dfrac{dy}{dx} + y = 3^x.$

b) $(y + y')^2 = \arctan(x)$

c) $\dfrac{d^4y}{dx^4} - 6\dfrac{d^2y}{dx^2} = 0$

2) (12 points each) Find all singular points for the following differential equations and classify them as regular or irregular.

a) $(x^2 - 2x)y'' + xy' + x^2y = 0$

b) $(x^2 - x - 20)^2y'' + (6x - 30)^2y' + (x - 5)y = 0$

3) A tank holding 400 L of brine initially contains 20 kg of dissolved salt. The tank has two valves, A and B, only one of which is open at any given time. When valve A is open, 10L/min of brine flows into the tank containing .04kg/L of salt. When valve B is open, 10L/min of brine flows into the tank containing .15kg/L of salt. The exit valve C removes 10L/min of brine. Let $x(t)$ denote the amount of salt in the tank at time t and suppose valve A is initially open.

a) (7 points) Determine an initial value problem for $x(t)$ if only valve A is ever open.

b) (4 points) How does your initial value problem from a) change if, after 15 minutes, valve B is opened and valve A closed? Write down the change explicitly.

4) (14 points) Solve the initial value problem

$$\frac{dx}{dt} = \frac{2}{5} - \frac{x(t)}{40}$$

if $x(0) = 20$.

5) Suppose you are given a mass-spring oscillator initially at rest 1 m below equilibrium. Suppose the mass m is 3 kg, the damping coefficient is zero, and the spring constant is $k = 27$. Let $y(t)$ denote the position of the mass.

a) (7 points) Determine an initial value problem for $y(t)$ if there are no external forces acting on the system.

b) (2 points) How does your initial value problem from a) change if there is an external force of $5\cos(4\pi t)$ N acting on the system? Write down the change explicitly.

c) (4 points) How does your initial value problem from a) change if, at $t = \pi/4$ seconds, the mass is struck by a hammer with a force of 10 N? Write down the change explicitly.

6) (22 points) Solve the formal initial value problem

$$3y'' + 27y = 10\delta\left(t - \frac{\pi}{4}\right)$$

if $y(0) = 1$ and $y'(0) = 0$.

7) Consider a simple RL circuit with one voltage source, one resistor, and one inductor, where the value of the resistance is 10 Ω and the value of the inductance is $8/5$ H. Let $I(t)$ denote the current and suppose that $I(0) = 2$ A and $I'(0) = 0$.

a) (5 points) Determine a differential equation whose solution yields $I(t)$ if the total voltage $E(t)$ of the system is zero.

b) (2 points) How does your equation from a) change if the total voltage of the system is given by $E(t) = \sin(4t)$? Write down the change explicitly.

c) (10 points) Suppose the circuit from part b) also contains a capacitor with capacitance $1/10$ F. Determine a differential equation whose solution yields $I(t)$.

8) (38 points) Solve the initial value problem

$$\frac{8}{5}I'' + 10I' + 10I = 4\cos(4t)$$

$I(0) = 2,\ I'(0) = 0.$

9) (13 points) Let u denote the temperature inside the cylinder $x^2 + y^2 \leq 9$. Suppose solutions $u = u(r, \theta, z, t)$ are assumed independent of z and θ (so in fact, $u = u(r, t)$). The Heat Equation is

$$k \frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right);$$

Find two ordinary differential equations that, when solved, will produce solutions to the Heat Equation, but DO NOT SOLVE THE EQUATIONS. *Hint:* begin by assuming $u(r, t) = f(r)g(t)$ for some twice-differentiable functions f and g.

10) (39 points) If α is a real number, solve the differential equations

$$g'(t) + \alpha g(t) = 0$$

and

$$r^2 f''(r) + r f'(r) + \alpha r^2 f(r) = 0$$

given that $f(3)g(t) = 1$.

BONUS 1: (10 points) Suppose that f_1 and f_2 are two functions which are analytic at the point $x = 0$ with radii of convergence r_1 and r_2, respectively. If r_1 and r_2 are both larger than zero, determine the minimum radius of convergence of $f_1 + f_2$.

BONUS 2: (10 points) Find the inverse Laplace Transform of

$$F(s) = \arctan\left(\frac{1}{s}\right).$$

LaPlace Transforms and Properties

1. $\mathcal{L}(e^{at})(s) = \dfrac{1}{s-a},\ s > a$

2. $\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}},\ s > 0$

3. $\mathcal{L}(\cos(at))(s) = \dfrac{s}{s^2+a^2},\ s > 0$

4. $\mathcal{L}(\sin(at))(s) = \dfrac{a}{s^2+a^2},\ s > 0$

5. $\mathcal{L}(\delta(t-a))(s) = e^{-as},\ s > 0$ where δ is the Dirac delta “function”.

6. $\mathcal{L}(u(t-a)f(t-a))(s) = e^{-as}\mathcal{L}(f)(s)$ where u is the Heaviside function.

7. $\mathcal{L}(e^{at}f(t))(s) = \mathcal{L}(f)(s-a)$

For number 8): a) If you choose variation of parameters to solve this problem,

$$\int e^{at}\cos(bt)\ dt = \frac{e^{at}(a\cos(bt) + b\sin(bt))}{a^2+b^2}.$$

b) If you choose to apply the Laplace Transform

$$\frac{(5/2)s}{(s+5)(s+5/4)(s^2+16)} = \frac{10}{123(s+5)} - \frac{40}{843(s+5/4)} - \frac{10(39s-400)}{11521(s^2+16)}.$$