Math 454/554 Assignment 1 Solutions

1) .

a) $\tan(x + \frac{x^2}{2} + C)$

b) $\frac{\sin(x)}{x^2}$

c) $7e^{-2x-2} + 5xe^{-2x-2}$

d) $(-\ln(x) + C_1)e^{-2x} + (-\frac{1}{x} + C_2)xe^{-2x}$

e) The relations between the coefficients are

$$a_0 + a_1 + 2a_2 = 0$$

and for $n \geq 2$,

$$a_{n-2} + (n+1) \cdot na_{n+1} + n^2 a_n + a_{n-1} = 0.$$

2) $D_1 = C_1 + C_2$, $D_2 = i(C_1 - C_2)$

3)

$$\begin{aligned}\int_0^\pi \cos(mx)\cos(nx)\ dx &= \frac{1}{2}\int_0^\pi \cos((m-n)x) + \cos((m+n)x)\ dx \\ &= \frac{1}{2}\left\{\frac{\sin((m-n)x)}{m-n} + \frac{\sin((m+n)x)}{m+n}\right\}\Bigg|_0^\pi\end{aligned}$$

which is legitimate since $n \neq m$ implies that $m-n$ is never 0. Substituting the limits of integration, we get

$$\frac{1}{2}\left\{\frac{\sin((m-n)\pi)}{m-n} + \frac{\sin((m+n)\pi)}{m+n}\right\} - 0 = 0.$$

4) If $x = 1$ or $x = 0$, then $f_n(x) = 0$ for all n. If $0 < x < 1$, then $0 < x^2 < 1$ and so $0 < 1 - x^2 < 1$. Using l'Hopital's rule, if we fix such an x,

$$\begin{aligned}\lim_{n\to\infty} n^2 x(1-x^2)^n &= x \lim_{n\to\infty} \frac{n^2}{(1-x^2)^{-n}} = x \lim_{n\to\infty} \frac{2n}{-\ln(1-x^2)(1-x^2)^{-n}} \\ &= x \lim_{n\to\infty} \frac{2}{(\ln(1-x^2))^2(1-x^2)^{-n}} = \frac{2x}{(\ln(1-x^2))^2} \lim_{n\to\infty}(1-x^2)^n \\ &= 0\end{aligned}$$

since $0 < 1 - x^2 < 1$. Therefore, $f_n \to 0$ pointwise.

However,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 f_n(x)\ dx &= \lim_{n\to\infty} \int_0^1 n^2 x(1-x^2)^n\ dx = \lim_{n\to\infty} n^2 \int_0^1 x(1-x^2)^n\ dx \\
&= \lim_{n\to\infty} -n^2 \frac{(1-x^2)^{n+1}}{2(n+1)}\Big|_0^1 = \lim_{n\to\infty} \frac{n^2}{2(n+1)} \\
&= \infty
\end{aligned}$$

and so

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = \infty \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\ dx.$$

If $f_n \to 0$ uniformly, these two quantities would be equal, and so we conclude that f_n does not converge to 0 uniformly.

5)

$$\begin{aligned}
&\int_a^b \int_a^b (f(x)g(y) - g(x)f(y))^2\ dx\ dy \\
&= \int_a^b \int_a^b (f(x)^2 g(y)^2 - 2f(x)g(y)g(x)f(y) + g(x)^2 f(y)^2)\ dx\ dy \\
&= \int_a^b f(x)^2\ dx \int_a^b g(y)^2\ dy - 2\int_a^b f(x)g(x)\ dx \int_a^b f(y)g(y)\ dy + \int_a^b g(x)^2\ dx \int_a^b f(y)^2\ dy \\
&= \|f\|_2^2 \|g\|_2^2 - 2\langle f, g\rangle^2 + \|f\|_2^2 \|g\|_2^2 \\
&= 2\|f\|_2^2 \|g\|_2^2 - 2\langle f, g\rangle^2.
\end{aligned}$$

Dividing by two on either side, we get the desired result. Now observe that since $(f(x)g(y) - g(x)f(y))^2 \geq 0$,

$$0 \leq \frac{1}{2} \int_a^b \int_a^b (f(x)g(y) - g(x)f(y))^2\ dx\ dy = \|f\|_2^2 \|g\|_2^2 - \langle f, g\rangle^2,$$

so that

$$\langle f, g\rangle^2 \leq \|f\|_2^2 \|g\|_2^2.$$

Taking square roots,

$$|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2.$$

6) If $x = 0$, then $f_n(0) = 1$ for all n. Now if $x \in [-1, 1] \setminus \{0\}$, then there exists an $N \in \mathbb{N}$ with

$$\frac{1}{N} < |x|.$$

Then for all $n \geq N$, $f_n(x) = 0$, which implies that the pointwise limit of $\{f_n\}_{n=1}^{\infty}$ is f where for $x \in [-1, 1]$,

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Since f is discontinuous at $x = 0$, this shows that the pointwise limit is not a continuous function. However, $f_n \to 0$ in mean since

$$\begin{aligned} \int_{-1}^{1} (f_n(x))^2 \, dx &= \int_{-1/n}^{1/n} (1 - |nx|)^2 \, dx \\ &= 2 \int_{0}^{1/n} (1 - nx)^2 \, dx \\ &= 2 \int_{0}^{1/n} 1 - 2nx + n^2x^2 \, dx \\ &= 2 \left(x - nx^2 + \frac{n^2x^3}{3} \right) \Big|_0^{1/n} \\ &= 2 \left(\frac{1}{n} - \frac{1}{n} + \frac{1}{3n} \right) = \frac{2}{3n} \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $f_n \to 0$ in mean.