Math 454/554 Assignment 2

Due Thursday, 9/30

1) Show that, for $f \in C([a,b])$, the mapping $\|\cdot\|_\infty : C([a,b]) \to [0,\infty)$ given by

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

is a norm. You may freely use the triangle inequality for real numbers and the fact that a continuous function on a closed interval attains its maximum.

2) (#4(a), Section 58) For all nonnegative integers n and m and for all $c > 0$, establish that

$$\int_{-c}^{c} \sin\left(\frac{m\pi x}{c}\right)\cos\left(\frac{n\pi x}{c}\right)\,dx = 0.$$

3) If $\gamma_m(x) = \sqrt{\frac{2}{c}}\sin\left(\frac{m\pi x}{c}\right)$ for m a natural number, compute the Fourier coefficients $\langle 1, \gamma_m\rangle$. The integral is from $x = 0$ to $x = c$.

4) Prove the parallelogram property for $(C([a,b]), \|\cdot\|_2)$; that is, show

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2$$

for all $f, g \in C([a,b])$. Does the same equality hold with $(C([a,b]), \|\cdot\|_\infty)$? Prove or give a counterexample.

5) (#7 (a), Section 18) Let $s_N(x) = \sum_{n=0}^{N} \frac{\sin((2n+1)x)}{2n+1}$ for $0 < x < \pi$. Using the equality

$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$$

and summing both sides of this equality from $n = 0$ to $n = N$, derive the formula

$$2\sum_{n=0}^{N} \cos((2n+1)x) = \frac{\sin(2(N+1)x)}{\sin(x)}.$$

Due Thursday 10/7

6) Prove that $\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k$ can be expressed as a trigonometric polynomial for all natural numbers k; i.e., show that $\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k$ can be written as a finite linear combination from the list $\{\cos(\frac{n\pi x}{c}), \sin(\frac{m\pi x}{c})\}$ where m is a natural number and n is a nonnegative integer. *Hint:* Mathematical induction.

7) For a real number $p \geq 1$ and $f \in C([0,1])$, define the p-norm of f as

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Recall the definition $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$. Show that

$$\|f\|_p \leq \|f\|_\infty$$

for all $p \geq 1$, thus in some sense justifying the use of the notation $\|f\|_\infty$.

EXTRA CREDIT: Show that, in the previous problem, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. You may assume that $\|f\|_p$ increases as $p \to \infty$ for all f.

8) a) With $f, g \in C([a,b])$, show that $M_g : (C([a,b]), \|\cdot\|_\infty) \to (C([a,b]), \|\cdot\|_\infty)$ defined by $M_g(f) = fg$ is a linear map.

b) Prove that M_g as defined in part a) is continuous. *Hint:* It will suffice to show that $f_n \to f$ in $\|\cdot\|_\infty$ implies that $gf_n \to fg$ in $\|\cdot\|_\infty$.

9) (#2 Section 30) (Heisenberg's uncertainty principle- almost) Use the linear operators $T = M_x$ and $S = \frac{\partial}{\partial x}$ to illustrate the fact that products ST and TS are not always the same.

10) (#4 Section 30) Show that each of the functions

$$y_1 = \frac{1}{x} \text{ and } y_2 = \frac{1}{1+x}$$

satisfies the *nonlinear* differential equation

$$y' + y^2 = 0.$$

Then show that the sum $y_1 + y_2$ fails to satisfy that equation. Also show that if c is a constant, where $c \neq 0$ and $c \neq 1$, neither cy_1 nor cy_2 satisfies the equation.

11) (only mandatory for graduate students) On $[0, c]$, calculate the Fourier cosine and sine series coefficients for $f(x) = 1 - \frac{x}{c}$. Run the program `fseries` in MatLab, for example, with data choice (2) for both the cosine (choose the even extension of f) and sine (choose the odd extension of f) series. Do this for $N = 8, 16$, and 32 in both series. Which series appears to converge to f faster in mean?