Math 454/554 Assignment 2 Solutions

1)

- $\|f\|_\infty = 0$ if and only if $f = 0$: If $f = 0$, then $\|f\|_\infty = \max_{x\in[a,b]} |f(x)| = 0$. Conversely, if $\|f\|_\infty = 0$, then $|f(x_0)| \le \max_{x\in[a,b]} |f(x)| = \|f\|_\infty = 0$ for all $x_0 \in [a,b]$, hence $f(x_0) = 0$ and so $f = 0$.

- $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$ for all scalars α:

  $$\|\alpha f\|_\infty = \max_{x\in[a,b]} |\alpha f(x)| = \max_{x\in[a,b]} |\alpha| \cdot |f(x)| = |\alpha| \max_{x\in[a,b]} |f(x)| = |\alpha|\|f\|_\infty.$$

- $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$: For all $f, g \in C([a,b])$ and $x_0 \in [a,b]$, we have by the triangle inequality for real numbers,

  $$|f(x_0)+g(x_0)| \le |f(x_0)|+|g(x_0)| \le \max_{x\in[a,b]} |f(x)|+\max_{x\in[a,b]} |g(x)| = \|f\|_\infty+\|g\|_\infty.$$

  Now since f and g are continuous on $[a,b]$, there is an $x_0 \in [a,b]$ with $\max_{x\in[a,b]} |f(x)+g(x)| = |f(x_0)+g(x_0)|$ (this result is not necessary to obtain the conclusion). Therefore,

  $$\|f\|_\infty = \max_{x\in[a,b]} |f(x)| = |f(x_0)| \le \|f\|_\infty + \|g\|_\infty.$$

2) If $m = n \ge 1$, then using the trig identity $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$,

$$\begin{aligned}
\int_{-c}^{c} \sin\left(\frac{n\pi x}{c}\right)\cos\left(\frac{n\pi x}{c}\right)\, dx &= \frac{1}{2}\int_{-c}^{c} \sin\left(\frac{2n\pi x}{c}\right) + \sin(0)\, dx \\
&= \frac{1}{2}\int_{-c}^{c} \sin\left(\frac{2n\pi x}{c}\right)\, dx \\
&= -\frac{c}{4n\pi}\cos\left(\frac{2n\pi x}{c}\right)\Big|_{-c}^{c} \\
&= -\frac{c}{4n\pi}(\cos(2n\pi) - (\cos(-2n\pi)) \\
&= -\frac{c}{4n\pi}(\cos(2n\pi) - (\cos(2n\pi)) \\
&= 0
\end{aligned}$$

since cosine is an even function.

If $m \neq n$, then again using $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B) + \sin(A-B))$,

$$\int_{-c}^{c} \sin\left(\frac{m\pi x}{c}\right)\cos\left(\frac{n\pi x}{c}\right)\ dx = \frac{1}{2}\int_{-c}^{c} \sin\left(\frac{(m+n)\pi x}{c}\right) + \sin\left(\frac{(m-n)\pi x}{c}\right)\ dx$$
$$= -\frac{1}{2}\left(\frac{c}{(m+n)\pi}\cos\left(\frac{(n+m)\pi x}{c}\right) + \frac{c}{(m-n)\pi}\cos\left(\frac{(m-n)\pi x}{c}\right)\right)\Big|_{-c}^{c}$$
$$= -\frac{c}{2(m+n)\pi}\left(\cos((n+m)\pi) - \cos(-(n+m)\pi)\right)$$
$$- \frac{c}{2(m-n)\pi}\left(\cos((n-m)\pi) - \cos(-(n-m)\pi)\right)$$
$$= -\frac{c}{2(m+n)\pi}\left(\cos((n+m)\pi) - \cos((n+m)\pi)\right)$$
$$- \frac{c}{2(m-n)\pi}\left(\cos((n-m)\pi) - \cos((n-m)\pi)\right)$$
$$= 0$$

3)

$$\begin{aligned}\langle 1, \gamma_m\rangle &= \sqrt{\frac{2}{c}}\int_0^c \sin\left(\frac{m\pi x}{c}\right)\ dx = -\sqrt{\frac{2}{c}}\frac{c}{\pi m}\cos\left(\frac{m\pi x}{c}\right)\Big|_0^c \\ &= -\frac{\sqrt{2c}}{\pi m}(\cos(m\pi) - \cos(0)) = -\frac{\sqrt{2c}}{\pi m}(\cos(m\pi) - 1) \\ &= -\frac{\sqrt{2c}}{\pi m}((-1)^m - 1) \\ &= \begin{cases} 0 & m \text{ even} \\ \frac{4\sqrt{c}}{\sqrt{2}\pi m} & m \text{ odd} \end{cases}\end{aligned}$$

4) Expanding the left-hand side,

$$\begin{aligned}
\|f+g\|_2^2+\|f-g\|_2^2 &= \int_a^b (f(x)+g(x))^2\ dx + \int_a^b (f(x)-g(x))^2\ dx \\
&= \int_a^b f(x)^2\ dx + 2\int_a^b f(x)g(x)\ dx + \int_a^b g(x)^2\ dx \\
&+ \int_a^b f(x)^2\ dx - 2\int_a^b f(x)g(x)\ dx + \int_a^b g(x)^2\ dx \\
&= 2\int_a^b f(x)^2\ dx + 2\int_a^b g(x)^2\ dx \\
&= 2\|f\|_2^2 + 2\|g\|_2^2
\end{aligned}$$

For the second part of the question, if we take $f(x) = 1$ and $g(x) = x$ on $[-1, 1]$, then

$$\|f\|_\infty = 1, \|g\|_\infty = 1, \|f+g\|_\infty = 2, \|f-g\|_\infty = 2.$$

So

$$2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 2 + 2 = 4,$$

but

$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 4 + 4 = 8,$$

which implies that the property fails to hold in general for $\|\cdot\|_\infty$.

5) Multiplying both sides of the desired equality by $\sin(x)$, we obtain

$$\sum_{n=0}^{N} 2\sin(x)\cos((2n+1)x) = \sin(2(N+1)x).$$

Employing the trigonometric identity $2\sin(A)\cos(B) = \sin(A+B)+\sin(A-B)$, we get

$$\begin{aligned}
\sum_{n=0}^{N} 2\sin(x)\cos((2n+1)x) &= \sum_{n=0}^{N}(\sin((2n+2)x) + \sin(-2nx)) \\
&= \sum_{n=0}^{N}(\sin((2n+2)x) - \sin(2nx))
\end{aligned}$$

since sine is an odd function. This last is a telescoping series that reduces to the last term in the series, which is

$$\sin((2N+2)x) = \sin(2(N+1)x).$$

6) For $k = 1$, the result is evidently true. Inductively assume that $\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k$ can be expressed as a finite linear combination of trig polynomials. By a judicious use of zeros for coefficients, we may then suppose that there is an integer N and real numbers $A_0, \ldots, A_N$ and $B_1, \ldots, B_N$ with

$$\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k = A_0 + \sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\right).$$

Multiplying both sides of this equality by $\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}$,

$$\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^{k+1} = \left(A_0 + \sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\right)\right)\cdot\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)$$

$$= A_0\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right) + \frac{1}{2}\left(\sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\right)\right)$$

$$+\frac{1}{2}\left(\sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\right)\right)\cdot\left(\cos\left(\frac{\pi x}{c}\right)\right)$$

$$= A_0\left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right) + \frac{1}{2}\left(\sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\right)\right)$$

$$+\frac{1}{2}\left(\sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right)\cos\left(\frac{\pi x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\cos\left(\frac{\pi x}{c}\right)\right)\right).$$

The only part of this last expression which is not evidently a trig polynomial is

$$\sum_{n=1}^{N}\left(A_n \cos\left(\frac{\pi n x}{c}\right)\cos\left(\frac{\pi x}{c}\right) + B_n \sin\left(\frac{n\pi x}{c}\right)\cos\left(\frac{\pi x}{c}\right)\right). \tag{1}$$

However, using the trig identities $\sin(A)\cos(B) = \frac{1}{2}(\sin(A+B)+\sin(A-B))$ and $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B)+\cos(A-B))$, we have that

$$\cos\left(\frac{\pi n x}{c}\right)\sin\left(\frac{\pi x}{c}\right) = \frac{1}{2}\left(\sin\left(\frac{\pi(n+1)x}{c}\right)+\sin\left(\frac{\pi(n-1)x}{c}\right)\right)$$

and

$$\cos\left(\frac{\pi n x}{c}\right)\cos\left(\frac{\pi x}{c}\right) = \frac{1}{2}\left(\cos\left(\frac{\pi(n+1)x}{c}\right)+\cos\left(\frac{\pi(n-1)x}{c}\right)\right),$$

which implies that expression (1) may be written as

$$\sum_{n=1}^{N}\left(\frac{A_n}{2}\left(\cos\left(\frac{\pi(n+1)x}{c}\right)+\cos\left(\frac{\pi(n-1)x}{c}\right)\right)+\frac{B_n}{2}\left(\sin\left(\frac{\pi(n+1)x}{c}\right)+\sin\left(\frac{\pi(n-1)x}{c}\right)\right)\right)$$

and so is a trig polynomial. This ends the proof.

7) Recall again that for all $x \in [0,1]$,

$$|f(x)| \le \max_{x\in[0,1]} |f(x)| = \|f\|_\infty.$$

Hence, for all $1 \le p < \infty$, $|f(x)|^p \le \|f\|_\infty^p$, and so

$$\int_0^1 |f(x)|^p \, dx \le \int_0^1 \|f\|_\infty^p \, dx = \|f\|_\infty^p \int_0^1 \, dx = \|f\|_\infty^p.$$

Then taking p^{th} roots on both sides,

$$\|f\|_p = \left\{\int_0^1 |f(x)|^p \, dx\right\}^{\frac{1}{p}} \le \{\|f\|_\infty^p\}^{\frac{1}{p}} = \|f\|_\infty.$$

EXTRA CREDIT: Come see me if you are interested in a proof.

8) a) If f, g, and h are in $C([a,b])$ and α, β are scalars, then

$$M_g(\alpha f+\beta h) = g(\alpha f+\beta h) = g(\alpha f)+g(\beta h) = \alpha g f+\beta g h = \alpha M_g(f)+\beta M_g(h),$$

which establishes linearity of M_g.

b) Suppose $f_n \to f$ in $\|\cdot\|_\infty$. This means that for all $\varepsilon_0 > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f - f_n\|_\infty = \max_{x \in [a,b]} |f(x) - f_n(x)| < \varepsilon_0.$$

Then for all $x \in [a, b]$,

$$|M_g(f - f_n)(x)| = |g(x)(f(x) - f_n(x))| = |g(x)||f(x) - f_n(x)| \leq \|g\|_\infty |f(x) - f_n(x)|.$$

Taking the max on either side, we obtain

$$\|M_g(f - f_n)\|_\infty \leq \|g\|_\infty \|f - f_n\|_\infty.$$

Therefore if we choose $\varepsilon > 0$ and $\varepsilon_0 = \dfrac{\varepsilon}{\|g\|_\infty}$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|M_g(f - f_n)\|_\infty \leq \|g\|_\infty \|f - f_n\|_\infty < \|g\|_\infty \cdot \frac{\varepsilon}{\|g\|_\infty} = \varepsilon.$$

This shows that M_g is continuous in the norm $\|\cdot\|_\infty$.

9) If we choose $f(x) = 1$,

$$(ST)(f)(x) = (ST)(1) = S(x) = 1.$$

and

$$(TS)(f)(x) = (TS)(1) = T(0) = 0.$$

Therefore, $ST \neq TS$.

10) If $y(x) = \dfrac{1}{x}$,

$$y' + y^2 = -\frac{1}{x^2} + \frac{1}{x^2} = 0.$$

If $y(x) = \dfrac{1}{x+1}$,

$$y' + y^2 = -\frac{1}{(x+1)^2} + \frac{1}{(x+1)^2} = 0.$$

However, if $y(x) = \frac{1}{x} + \frac{1}{x+1}$,

$$y' + y^2 = -\frac{1}{x^2} - \frac{1}{(x+1)^2} + \frac{1}{x^2} + \frac{2}{x(x+1)} + \frac{1}{(x+1)^2} = \frac{2}{x(x+1)}$$

which is never equal to zero. Finally, if c is a real number and $y(x) = \frac{c}{x}$,

$$y' + y^2 = -\frac{c}{x^2} + \frac{c^2}{x^2} = \frac{c^2 - c}{x^2},$$

which is only zero if $c = 1$ or $c = 0$. Similarly, if $y(x) = \frac{c}{x+1}$,

$$y' + y^2 = -\frac{c}{(x+1)^2} + \frac{c^2}{(x+1)^2} = \frac{c^2 - c}{(x+1)^2},$$

which is again only zero if $c = 1$ or $c = 0$.