Math 454/554 Assignment 5

Due Thursday, 11/11

1) In this problems, you will construct a continuous function whose Fourier series diverges at the point $x = 0$. The construction can be modified to obtain divergence at any single point. For natural numbers p and q with $p \geq q$, define

$$t_{q,p}(x) = 2\sum_{k=1}^{q} \frac{\sin(px)\sin(kx)}{k} = \sum_{|k|\leq q, k\neq 0} \frac{\cos((p-k)x)}{k}$$

for $-\pi \leq x \leq \pi$.

Let $\alpha_k = \frac{1}{k^2}$, $p_k = 2^{k^3+1}$, and $q_k = 2^{k^3}$. Observe that $p_k \geq q_k$ for trivial reasons.

a) Show that $p_k + q_k < p_{k+1} - q_{k+1}$ for all integers $k \geq 1$.

b) Compute $\lim_{k\to\infty} |\alpha_k| \ln(q_k)$.

Now consider the function $f(x) = \sum_{k=1}^{\infty} \alpha_k t_{p_k,q_k}(x)$.

c) Use the Weierstrass M-test (see Section 16, p. 47) to show that the convergence is uniform in the definition for f, hence $f \in C([-\pi,\pi])$.

d) Show that the Fourier coefficients B_k for f corresponding to $\sin(kx)$ are all zero. *Extra Credit:* Calculate the Fourier cosine coefficients for f on the interval $[-\pi, \pi]$.

e) If S_n is the n^{th} Fourier cosine partial sum for f, it can be shown that

$$|S_{q_k+p_k}(0) - S_{p_k}(0)| = |\alpha_k| \sum_{m=1}^{q_k} \frac{1}{m}.$$

Using the fact that $\sum_{m=1}^{q_k} \frac{1}{m} \geq \int_1^{q_k} \frac{1}{x}\,dx$, show that

$$\lim_{k\to\infty} |S_{q_k+p_k}(0) - S_{p_k}(0)| = \infty,$$

which yields that the Fourier series of f diverges at $x = 0$.

2) # 1, Section 7

3) (Gibbs phenomenon) # 7, Section 18, parts b)-d)

4) Let f, g be continuous on the interval $[a, b]$ and possibly complex-valued. Recall the (potentially) complex-valued inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

a) If $h \in C([a, b])$ is real-valued, recall that $M_h(f) = hf$. Prove that

$$\langle M_h(f), g \rangle = \langle f, M_h(g) \rangle.$$

b) Suppose in addition that f and g are differentiable on $(-\infty, \infty)$ and that $f', g' \in C([a, b])$. Let $D(f)(x) = f'(x)$. Show that

$$\langle iD(f), g \rangle = \langle f, iD(g) \rangle$$

where $i = \sqrt{-1}$.