Math 454/554 Final

Directions: Due Wednesday, December 22nd. If you use separation of variables on any problem, be sure to cover all the cases for constants λ.

1) (15 points) a) Compute the Fourier series of the function $f(x) = \cos(\frac{3}{2}x)$ on the interval $[-\pi, \pi]$.

b) (5 points) Use the Weierstrass M-test to show that the series converges uniformly on the interval $[-\pi, \pi]$.

2) (20 points) Suppose transverse displacements z of a membrane stretched over the circle $\rho = 3$ on the xy-plane are independent of the spherical coordinate ϕ, so we may write $z = z(\rho, t)$. Solve the boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = 16\left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho}\right)$$

subject to

$$z(3,t) = \frac{\partial z}{\partial t}(\rho, 0) = 0, \qquad z(\rho, 0) = 1.$$

3) (30 points) Let $u(x,t)$ denote temperature in a homogeneous body on the interval $0 \le x \le 1$ for $t \ge 0$. Use the method of variation of parameters to solve the boundary value problem

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + t$$

and

$$\frac{\partial u}{\partial x}(0,t) = u(1,t) = u(x,0) = 0.$$

4) (15 points) For $0 \le x \le \pi$, consider the second-order homogeneous differential equation

$$f''(x) + \lambda f(x) = 0$$

Recall that this may be expressed as $Tf = -\lambda f$ where T is the linear map

$$T = D^2$$

and D is the derivative with respect to x. Recall also that if $u_1, \ldots, u_n$ is an orthonormal basis for $\mathbb{R}^n$, any linear map from $\mathbb{R}^n$ to itself may be written as a matrix A, with entries

$$a_{i,j} = Au_j \cdot u_j$$

where "$\cdot$" is the usual dot product.

a) (8 points) If $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, calculate the entries $T_{n,m} = \langle T\phi_m, \phi_n \rangle$ for $n, m \geq 1$. Thus, we represent T as an infinite matrix.

b) (7 points) Recall that a linear map S is *bounded* if $\|Sf\|_2 \leq C\|f\|_2$ for all $f \in C([0, \pi])$. Using the matrix representation of T, show that T is not bounded.

5) (15 points) For $x \in \mathbb{R}$, define the decimal part of x, $\langle x \rangle$, to be

$$\langle x \rangle = x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x.

a) By choosing a suitable interval, calculate a Fourier series for $f(x) = \langle x \rangle^2 - \langle x \rangle$. *Hint:* wouldn't it be nice if $\langle x \rangle = x$...

b) Using the series from part a), find a value for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.