Math 454/554 Midterm

IN-CLASS PORTION

1) a) For $u = u(x, y, z, t)$, define the Laplacian ∇^2 of u.

b) State the formula for the Laplacian in cylindrical coordinates. You do not need to show any work, just the formula.

2) Let $u = u(x, y, z, t)$ denote temperatures in a homogeneous body with no internal heat source or sink.

a) What does the phrase “steady-state” mean in terms of u and its partial derivatives?

b) What does the heat equation reduce to in the steady-state case?

c) Suppose u is independent of z, so that $u = u(x, y, t)$. In terms of u and its partial derivatives, what boundary condition ensues if the medium is insulated along the x-axis?

d) Suppose further that u is independent of y and z, so that $u = u(x, t)$. If $0 \leq x \leq \pi$ and $t \geq 0$, give an example of boundary conditions on the spatial variable x that will ensure a Fourier cosine series occurs in the solution for u.

3) a) What does the notation $C([a, b])$ stand for?

b) For f and g in $C([a, b])$, define $\langle f, g\rangle$ on the interval $[a, b]$.

c) On the interval $[0, \pi]$, give an example of two nonzero functions f and g with $\langle f, g\rangle = 0$, if such an example exists. Support your example with calculations.

4) a) For a given sequence $\{\phi_n\}_{n=1}^{\infty} \subset C([a,b])$, what does it mean for $\{\phi_n\}_{n=1}^{\infty}$ to be orthonormal?

b) State Bessel's Inequality for $f \in C([a,b])$ and an orthonormal sequence $\{\phi_n\}_{n=1}^{\infty} \subset C([a,b])$.

c) Give an example of an infinite orthonormal sequence in $C([0,c])$.

d) Use Bessel's Inequality (or any other method at your disposal) to demonstrate that for all $f \in C([0,c])$,

$$\lim_{n\to\infty} \int_0^c f(x) \cos\left(\frac{\pi n x}{c}\right)\, dx = 0.$$

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TAKE HOME PORTION

1) (16 points) Let $u(x,t)$ denote temperature in a homogeneous body on the interval $0 \le x \le 1$ for $t \ge 0$. If $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$

and

$$u(x,0) = x^2,\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0,$$

find the general solution for $u(x,t)$.

2) (16 points) Let $u(\rho,\phi)$ denote steady-state temperature in a homogeneous body for the cylindrical coordinates $1 \le \rho \le 9$ and $0 \le \phi \le \frac{\pi}{2}$. If $u(\rho,\phi)$ satisfies the heat equation

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = 0$$

and

$$u(\rho,0) = u(\rho,\frac{\pi}{2}) = u(1,\phi) = 0,\ u(9,\phi) = 13,$$

find the general solution for $u(\rho,\phi)$. The Cauchy-Euler transformation $\rho = e^s$, which converts

$$A\rho^2 y''(\rho) + B\rho y'(\rho) + Cy(\rho) = 0$$

into

$$A\frac{\partial^2 y}{\partial s^2} + (B-A)\frac{\partial y}{\partial s} + Cy = 0,$$

where A, B, and C are constants, may be useful to you.

3) a) (5 points) Given $f(x) = x$, compute the Fourier series for f on the interval $[-2,2]$.

b) (3 points) Show that at the point $x_0 = 2$, the Fourier series for $f(x) = x$ on $[-2,2]$ does not converge to $f(x_0)$.

c) (5 points) Prove that the Fourier series for $f(x) = x$ on the interval $[-2, 2]$ converges to f in mean. You may use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4) (5 points- and a bit more of a challenge) Let p and q be natural numbers, with $p \leq q$. Show that for all real numbers $0 < x < \pi$ and all choices of p and q,

$$\left| \sum_{p \leq n \leq q} e^{inx} \right| \leq \csc\left(\frac{x}{2}\right).$$

Recall that $e^{inx} = \cos(nx) + i\sin(nx)$ and that the absolute value of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$ (a,b are real numbers).