Math 454/554 Notes 12/9

0.1 Broadening Our Horizons

For $f \in C([-\pi,\pi])$, we have

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(\pi n x) + B_n \sin(\pi n x))$$

in mean. Recall that

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$$

and

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}.$$

Substituting, we obtain

$$\begin{aligned} f(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + B_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) - iB_n \left(\frac{e^{inx} - e^{-inx}}{2} \right) \right) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{A_n - iB_n}{2} \right) e^{inx} + \left(\frac{A_n + iB_n}{2} \right) e^{-inx} \right) \end{aligned}$$

in mean. Regarding $B_0 = 0$ and setting $C_n = \dfrac{A_n - iB_n}{2}$, we obtain

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

in mean.

It is then easy to observe that if $n \neq m$,

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = 0.$$

We may then recover

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

using orthogonality.

0.2 The Discrete Fourier Transform

Applications are in signal processing, quantum computation, mechanical analysis, stock analysis, neurobiology (MRI machines). You may also see it referred to as the Quantum Fourier Transform.

We shall only consider *finite* Fourier series, written as exponentials. As such, we will translate and always start at $n = 0$, writing

$$f(x) = \sum_{n=0}^{k-1} c_n e^{inx}.$$

Note that there are now no issues of convergence; this is just pointwise equality. By periodicity of f, we may work on the interval $[0, 2\pi]$ instead of $[-\pi, \pi]$.

Suppose then that we would like to evaluate f at equally spaced intervals of length $\frac{2\pi}{k}$. Later, we will discuss why one would want to do this. We obtain k numbers $(y_0, y_1, \ldots, y_{k-1}$ where $y_j = f(\frac{2\pi j}{k})$, for $1 \leq j \leq k-1$ and $y_0 = f(0)$.

Example: Let $k = 4$ and $f(x) = c_0 + c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{3ix}$. Then

$$y_0 = f(0) = c_0 + c_2 + c_1 + c_3; \qquad y_1 = f(\frac{\pi}{2}) = c_0 + ic_1 - c_2 - ic_3$$

$$y_2 = f(\pi) = c_0 - c_1 + c_2 - c_3; \qquad y_3 = f(\frac{3\pi}{2}) = c_0 - ic_1 - c_2 + ic_3.$$

Now suppose we are given $y_0, y_1, \ldots, y_{k-1}$. Can we find $c_0, c_1, \ldots, c_{k-1}$ with

$$f(x) = \sum_{n=0}^{k-1} c_n e^{inx}$$

and $y_j = f(\frac{2\pi j}{k})$? The answer is yes, by tedious elimination, but let's try a better way.

Example: Suppose $y_0 = 1$, $y_1 = 0$, $y_2 = 1$, and $y_3 = 0$. We then need c_0, c_1, c_2, and c_3 with

$$1 = c_0 + c_1 + c_1 + c_3; \qquad 0 = c_0 + ic_1 - c_2 - ic_3$$

$$1 = c_0 - c_1 + c_2 - c_3; \qquad 0 = c_0 - ic_1 - c_2 + ic_3.$$

Write the system of equations in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

If we could invert the 4×4 matrix, we'd have an easy way to solve for c_0, c_1, c_2, and c_3.

Notation: If $A = (a_{n,m})_{n,m=1}^k$ is an $k \times k$ matrix with complex coefficients, define $A^* = (b_{n,m})_{n,m=1}^k$ where

$$b_{n,m} = \overline{a_{m,n}}.$$

In words, you make the rows of A into columns, the nth row going to the nth column, then take complex conjugates of all entries.

Example within an Example: If $A = \begin{pmatrix} i & 2+3i \\ -5 & 7-9i \end{pmatrix}$, then

$$A^* = \begin{pmatrix} -i & -5 \\ 2-3i & 7+9i \end{pmatrix}.$$

Now let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

One can check that

$$A^*A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Therefore, if we want

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

multiplying both sides by A^*, we get

$$\begin{pmatrix} 4c_0 \\ 4c_1 \\ 4c_2 \\ 4c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Now all we need do is compute

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

Then by equating components, $c_0 = c_2 = \dfrac{1}{2}$ and $c_1 = c_3 = 0$.

Definition: (Discrete Fourier Transform) The *Discrete Fourier Transform* on a k-element set of complex numbers is given by the matrix $F_k = (a_{n,m})_{n,m=1}^{k}$ where

$$a_{n,m} = \omega_k^{(n-1)(m-1)}$$

and $\omega_k = e^{\frac{2\pi i}{k}}$.

Under scaling by $\frac{1}{\sqrt{k}}$, F_k is a *unitary* transformation (a complex matrix U is unitary if $U^*U = I$ where I is the identity matrix).

However, for a random $y_0, y_1, \ldots y_k$, the k^2 multiplications involved in the Discrete Fourier Transform might be nasty. For computational efficiency, we'd like fewer multiplications.

0.3 The Fast Fourier Transform (FFT)

Note: This is just an algorithm for computing the Discrete Fourier Transform, NOT a transform in its own right. The prescription here is lifted from

Strang's *Linear Algeba and Applications* for $k = 2^j$ where $j \geq 2$, simulating bits in a computer. Strang assures us that a minor modification will work for k a composite integer. For k prime, we have to do something completely different.

We want to find $y_0, y_1, \ldots, y_{k-1}$ given $c_0, c_1, \ldots, c_{k-1}$. First, we shall see a description of the algorithm in general, then work an example.

FFT Step 1: Apply the permutation matrix P_{2^j} that rearranges

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2^j-1} \end{pmatrix} \text{ as } \begin{pmatrix} c_0 \\ c_2 \\ \vdots \\ c_{2^j-2} \\ c_1 \\ c_3 \\ \vdots \\ c_{2^j-1} \end{pmatrix}.$$

We can check that

$$(P_{2^j})_{n,m} = \begin{cases} \delta_{n,\frac{1+m}{2}}, & \text{if } 1 \leq n \leq 2^{j-1} \\ \delta_{n,\frac{2^j-1+m}{2}}, & \text{if } 2^{j-1} < n \leq 2^j. \end{cases}$$

FFT Step 2: Apply $F_{2^{j-1}}$ to the vectors

$$c'' = \begin{pmatrix} c_0 \\ c_2 \\ \vdots \\ c_{2^j-2} \end{pmatrix} \text{ and } c''' = \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{2^j-1} \end{pmatrix}.$$

Set $F_{2^{j-1}}c'' = y''$ and $F_{2^{j-1}}c''' = y'''$. Solve for y'' and y''', then set as the column vector

$$y' = \begin{pmatrix} y'' \\ y''' \end{pmatrix}.$$

FFT Step 3: Reconstruct y from y' by $y = B_{2^j}y'$ where

$$B_{2^j} = \begin{pmatrix} I_{2^{j-1}} & D_{2^{j-1}} \\ I_{2^{j-1}} & -D_{2^{j-1}} \end{pmatrix}.$$

Here, $I_{2^{j-1}}$ is the $2^{j-1} \times 2^{j-1}$ identity matrix and $D_{2^{j-1}}$ is the diagonal matrix with

$$(D_{2^{j-1}})_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ \omega_{2^j}^{n-1} & \text{if } n = m. \end{cases}$$

In matrix notation, the FFT is

$$F_{2^j} = B_{2^j} \begin{pmatrix} F_{2^{j-1}} & 0 \\ 0 & F_{2^{j-1}} \end{pmatrix} P_{2^j}.$$

Observe that replacing F_{2^j} with two copies of $F_{2^{j-1}}$ gives 2^{j-1} multiplications instead of 2^{2j}. In applying the FFT, repeat steps 1 and 2 until you have descended to F_2, then solve and reconstruct. This will end up with about $j * 2^{j-1}$ multiplications instead of 2^{2j}. For j large, this is significant.

Example: Solve

$$F_4 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

using the FFT.

Step 1: Rearrange

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ as } \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}$$

by applying the matrix

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set $c'' = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $c''' = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Step 2: Solve $F_2 c'' = y''$ and $F_2 c''' = y'''$. We have

$$y'' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

and

$$y''' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}.$$

Set

$$y' = \begin{pmatrix} y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 6 \\ -2 \end{pmatrix}.$$

Step 3: Reconstruct y from y' by $B_4 y' = y$ where

$$B_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}.$$

Hence,

$$y = B_4 y' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -2-2i \\ -2 \\ -2+2i \end{pmatrix}.$$

0.4 The Fourier Transform

Suppose that $f \in C((-\infty, \infty))$, which is the vector space of all continuous functions. We may regard f as complex-valued. If

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty,$$

we define the *Fourier Transform* $\mathcal{F}$ of f to be

$$\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t) e^{-itx} \, dt.$$

If, in addition, $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$, we can check using Fubini's Theorem that

$$\int_{-\infty}^{\infty} |\mathcal{F}(f)(x)|^2 \, dx = \int_{-\infty}^{\infty} |f(x)|^2 \, dx;$$

that is, the Fourier Transform is an isometry on the L^2 level. There are other versions that differ by the addition of a few constants in strategic places.

The Fourier Transform of f is also commonly written as $\hat{f}$, a notation which we shall now adopt. In fact, using the same definition, we can extend the Fourier Transform to functions which are not everywhere continuous, as the following example shows.

Example: Let

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if} |x| > 1. \end{cases}$$

It is immediate that $\hat{f}(0) = 2$. Then if $x \neq 0$,

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(x) e^{-itx} \, dt = \int_{-1}^{1} e^{-itx} \, dt = \frac{e^{-itx}}{-ix}\Big|_{-1}^{1} = \frac{e^{itx} - e^{-itx}}{ix} = \frac{2\sin(x)}{x}.$$

Note that $\lim_{x \to 0} \frac{2\sin(x)}{x} = 2$, so that we have "fixed" the discontinuity at $x = 0$. Therefore, the Fourier Transform of a discontinuous function can be continuous.

As an application, we can then compute an impossible-looking integral using the fact that the Fourier transform is an isometry on the L^2 level. Again let

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if} |x| > 1. \end{cases}$$

Then

$$\int_{x \neq 0} \frac{\sin^2(x)}{x^2} \, dx = \int_{x \neq 0} \frac{1}{4} |\hat{f}(x)|^2 \, dx = \frac{1}{4} \int_{x \neq 0} |f(x)|^2 \, dx = \frac{1}{4} \int_{-1}^{1} 1 \, dx = \frac{1}{2}.$$

0.5 The Nyquist Theorem

This theorem is also attributed to Shannon. For simplicity, we assume $f \in C((-\infty, \infty))$, though this assumption is not necessary. Suppose $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$.

Definition: f is *band-limited* with bandwidth B if there is a real number B such that

$$\hat{f}(x) = 0 \text{ if } |x| > B.$$

The Nyquist Theorem: Let f be band-limited with bandwidth B. Then f is completely determined by the values

$$f\left(\frac{n}{2B}\right)$$

where $n \in \mathbb{Z}$, $|n| \leq B$.

The Nyquist Theorem is the tool by which we pass from continuous to discrete data. A later version of these notes may include the proof.