Math 454/554 Notes

Recall the heat equation in one temporal and one spatial variable:

$$u_t = ku_{xx}$$

where k is a positive constant, $0 \leq x \leq c$, and $c \neq 0$. We also have the boundary conditions $u_x(0,t) = u_x(c,t) = 0$ and $u(x,0) = f(x)$ for some nonconstant function f. Disregarding the value of f, we found that the family of functions

$$\{A_0 + e^{-\frac{n^2\pi^2}{c^2}t} A_n \cos\left(\frac{\pi n x}{c}\right)\}_{n\in\mathbb{N}}$$

satisfies both the heat equation and the other boundary conditions. We also observed that any sum

$$A_0 + \sum_{n=1}^{N} e^{-\frac{n^2\pi^2}{c^2}t} A_n \cos\left(\frac{\pi n x}{c}\right)$$

also satisfies as well. Applying the remaining boundary condition, we see that

$$f(x) = A_0 + \sum_{n=1}^{N} A_n \cos\left(\frac{\pi n x}{c}\right).$$

Now we let m be a nonnegative integer and multiply both sides of this equation by $\cos\left(\frac{\pi m x}{c}\right)$. Integrating the result, we have that

$$\int_0^c f(x)\cos\left(\frac{\pi m x}{c}\right)\,dx = A_0\int_0^c \cos\left(\frac{\pi m x}{c}\right)\,dx + \sum_{n=1}^{N}\int_0^c A_n \cos\left(\frac{\pi n x}{c}\right)\cos\left(\frac{\pi m x}{c}\right)\,dx.$$

Now $\int_0^c \cos\left(\frac{\pi m x}{c}\right)\,dx = 0$ unless $m = 0$, in which case the value of the integral is c. This result tells us that

$$A_0 = \frac{1}{c}\int_0^c f(x)\,dx.$$

Note that if $n = m$,

$$\int_0^c \cos^2\left(\frac{\pi m x}{c}\right)\,dx = \int_0^c \frac{1+\cos\left(\frac{2m\pi x}{c}\right)}{2}\,dx = \left(\frac{x}{2} + \frac{c\sin\left(\frac{2m\pi x}{c}\right)}{4m\pi}\right)\Big|_0^c = \frac{c}{2}$$

One of your homework problems is to show that if $n \neq m$,

$$\int_0^c A_n \cos\left(\frac{\pi n x}{c}\right) \cos\left(\frac{\pi m x}{c}\right)\, dx = 0.$$

Therefore, if $n > 0$,

$$A_n = \frac{2}{c} \int_0^c f(x) \cos\left(\frac{\pi m x}{c}\right)\, dx.$$

Question: What if $A_n \neq 0$ for infinitely many n?

Even a very simple example like $f(x) = x$ has this property. Then no finite sum $A_0 + \sum_{n=1}^N A_n \cos\left(\frac{\pi n x}{c}\right)$ can be equal to $f(x)$ since for $m > N$,

$$\int_0^c \left(A_0 + \sum_{n=1}^N A_n \cos\left(\frac{\pi n x}{c}\right)\right) \cos\left(\frac{\pi m x}{c}\right)\, dx = 0.$$

Therefore, we are forced to consider an infinite sum! The only question is what is meant by the expression

$$A_0 + \sum_{n=1}^\infty A_n \cos\left(\frac{\pi n x}{c}\right),$$

i.e., in what sense does the sum converge? There are three different kinds of convergence we will discuss.

1. *Pointwise:* $f_n \to f$ pointwise on $[a,b]$ if for all $\varepsilon > 0$ and each $x \in [a,b]$, there is an $N = N(x, \varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon$ when $n \geq N$.

2. *Uniform:* $f_n \to f$ uniformly on $[a,b]$ if for all $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon$ when $n \geq N$ and for all $x \in [a,b]$. That is, the value N is independent of the choice of x.

3. *Mean (or L^2):* $f_n \to f$ in mean (or L^2) if for all $\varepsilon > 0$, there is an N such that

$$\int_a^b |f_n(x) - f(x)|^2\, dx < \varepsilon^2$$

for all $n \geq N$.

Note that uniform convergence is the strongest and implies convergence both pointwise and in mean. However, we will see examples of sequences of continuous functions on a closed interval that converge pointwise but not uniformly, that converge in mean but not pointwise (to the same function), and that converge pointwise but not in mean. You'll find examples of the first two on the homework, and we'll address the last later. Pointwise convergence, though nice, is too weak in general to give us what we want, so we will concentrate on the other two types of convergence.

First, since we would like $A_0 = \frac{1}{c}\int_0^c f(x)\ dx$ to exist, it makes sense to assume that f is continuous on $[0, c]$ since then the integral exists. We will weaken this hypothesis a bit in the future, though. Now the set of all continuous, real-valued functions on $[0, c]$ is a vector space over $\mathbb{R}$ since the sum of two continuous functions is again continuous and any multiple of a continuous function is continuous. Denote this vector space by $C([0, c])$.

Recall (or see for the first time) the following facts about uniform convergence. Suppose $\{f_n\}_{n\in\mathbb{N}}$ and f are in $C([0, c])$. Set $s_N(x) = \sum_{n=1}^N f_n(x)$. If $s_N \to f$ uniformly, then:

- If g is in $C([0, c])$, then $s_N g \to fg$ uniformly on $[0, c]$;
- $\displaystyle\lim_{N\to\infty}\int_0^c s_N(x)\ dx = \int_0^c \lim_{N\to\infty} s_N(x)\ dx = \int_0^c f(x)\ dx$

Without the hypothesis of uniform convergence, we have

- If f_n is differentiable on $(0, c)$ for all n and $\{s_N'\}_{N=1}^\infty$ converges uniformly, then if $s_N(x_0) \to f(x_0)$ for a single x_0 in $[0, c]$, $\displaystyle\lim_{N\to\infty} s_N'(x) = f'(x)$ for all $0 < x < c$.

So if we can then prove that $s_N(x) = A_0 + \sum_{n=1}^N A_n \cos\left(\frac{\pi n x}{c}\right)$ converges to f uniformly, then $A_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{\pi m x}{c}\right)\ dx$ for $n > 0$, and we are justified in writing

$$f(x) = A_0 + \sum_{n=1}^\infty A_n \cos\left(\frac{\pi n x}{c}\right).$$

It will then be the case that if $f_n' \to f'$ uniformly,

$$u(x, t) = A_0 + \sum_{n=1}^\infty e^{-\frac{n^2\pi^2}{c^2}t} A_n \cos\left(\frac{\pi n x}{c}\right)$$

is well-defined and satisfies the heat equation with the boundary conditions we have imposed. With some additional assumptions on the function f, we will obtain uniform convergence at a further point in the course. Note that WITHOUT these conditions, it is not always even true that $s_N \to f$ pointwise on $[0, c]$. However, $s_N \to f$ in mean regardless, which is one of many reasons why we consider this type of convergence.

Using uniform and mean convergence, we can define two *norms* on $C([0, c])$. Recall that a norm would be a map $\|\cdot\|$ from $C([0, 1])$ to the nonnegative real numbers satisfying, for all f and g in $C([0, 1])$,

- $\|f\| = 0$ if and only if $f = 0$,
- $\|\alpha f\| = |\alpha| \cdot \|f\|$ for $\alpha \in \mathbb{R}$,
- $\|f + g\| \leq \|f\| + \|g\|$ (triangle inequality).

We define $\|f\|_\infty = \max_{x\in[0,c]} |f(x)|$ and $\|f\|_2 = \left\{ \int_0^c |f(x)|^2 \, dx \right\}^{\frac{1}{2}}$. The continuity of f plus the compactness of $[0, c]$ guarantees that the max in the first quantity exists and that the integral in the second quantity is finite.

That the properties of a norm are satisfied by $\|\cdot\|_\infty$ is more or less immediate. It is a neat fact that $C([0, c])$ with this norm is both complete (every Cauchy sequence converges) and closed (every convergent sequence from $C([0, c])$ has its limit in $C([0, c])$). The latter follows from the properties of uniform continuity. In $\|\cdot\|_2$, it is certainly not true that the limit of continuous functions is again continuous- see the homework for a counterexample.

The only nontrivial part of verifying that $\|\cdot\|_2$ is a norm is to show $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$. This goes by the name *Minkowski's Inequality.* It will be a bit easier to prove after defining the inner product, $\langle f, g\rangle$, as follows:

$$\langle f, g\rangle = \int_0^c f(x)g(x) \, dx.$$

Note that $\|f\|_2^2 = \langle f, f\rangle$. Using this,

$$\|f+g\|_2^2 = \int_0^c (f+g)^2 \, dx = \int_0^c f(x)^2+2f(x)g(x)+g(x)^2 \, dx = \|f\|_2^2+2\langle f, g\rangle+\|g\|_2^2$$

and

$$(\|f\|_2 + \|g\|_2)^2 = \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2,$$

which tell us we only need to show that $\langle f, g\rangle \leq \|f\|_2\|g\|_2$. You will establish a (slightly) stronger inequality

$$|\langle f, g\rangle| \leq \|f\|_2\|g\|_2,$$

called the *Cauchy-Schwarz* inequality, in your homework. This will then show

$$\|f+g\|_2^2 \leq (\|f\|_2 + \|g\|_2)^2$$

and Minkowski's Inequality will be satisfied for $\|\cdot\|_2$ by taking square roots.