Math 454/554 Notes 9/28

Damage Control: Contrary to what was stated towards the end of class last time, the functions $\cos\left(\frac{\pi n x}{c}\right)$ and $\sin\left(\frac{\pi n x}{c}\right)$ are not always orthogonal on $[0, c]$. For example,

$$\int_0^\pi \sin(2x)\cos(2x)\ dx = \frac{1}{2}\int_0^\pi \sin(4x)\ dx$$
$$\left(\frac{\cos(4x)}{4}\right)\Big|_0^\pi$$
$$= 0$$

but

$$\begin{aligned}\int_0^\pi \sin(2x)\cos(x)\ dx &= \frac{1}{2}\int_0^\pi \sin(3x)+\sin(x)\ dx\\ &= \frac{1}{2}\left(\frac{-\cos(3x)}{3} - \cos(x)\right)\Big|_0^\pi\\ &= \frac{4}{3}\end{aligned}$$

In fact, the set $\{\phi_n\}_{n=0}^\infty$ is a complete orthonormal set on $[0, c]$. We will prove this in a circuitous fashion. First, we expand our scope to the interval $[-c, c]$ and observe that the system $\{\Phi\}_{n=0}^\infty \cup \{\Gamma_m\}_{m=1}^\infty$ defined on $[-c, c]$ by

$$\Phi_0(x) = \frac{1}{\sqrt{2c}}$$

$$\Phi_n(x) = \frac{1}{\sqrt{c}}\cos\left(\frac{\pi n x}{c}\right),\ n \geq 1$$

$$\Gamma_n(x) = \frac{1}{\sqrt{c}}\sin\left(\frac{\pi m x}{x}\right)$$

is indeed orthonormal. Calculations similar to what we obtained in the $[0, c]$ case show that $\{\Phi_n\}_{n=0}^\infty$ and $\{\Gamma_m\}_{m=1}^\infty$ are each orthonormal sets, and this time, your homework WILL show that

$$\int_{-c}^{c} \sin\left(\frac{m\pi x}{c}\right)\cos\left(\frac{n\pi x}{c}\right)\ dx = 0.$$

for all natural numbers m and all nonnegative integers n. We then arrive at a definition.

Definition: The *Fourier Series* of a continuous function f on the interval $[-c, c]$ is the series

$$a_0 + \sum_{n=1}^{\infty}(a_n \Phi_n + b_n \Gamma_n)$$

where $a_n = \langle f, \Phi_n \rangle$ and $b_n = \langle f, \Gamma_n \rangle$ and the convergence is in mean.

Written another way, this is

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{\pi n x}{c}\right) + B_n \sin\left(\frac{\pi n x}{c}\right)\right)$$

where $A_0 = 2a_0$ and $A_n = \frac{1}{\sqrt{c}}a_n$, $B_n = \frac{1}{\sqrt{c}}b_n$ for $n \in \mathbb{N}$.

We shall then employ the following theorem to show that the Fourier orthonormal system is complete on $[-c, c]$.

Completeness Theorem: Let $\{\psi_n\}_{n=0}^{\infty}$ be an orthonormal sequence in $C([a, b])$. The following are equivalent:

- For every $\varepsilon_0 > 0$ and all $f \in C([a, b])$, there exists a g in the linear span of the ψ_n's with
$$\|f - g\|_2 < \varepsilon_0.$$
- $\sum_{n=0}^{\infty}\langle f, \psi_n \rangle^2 = \|f\|_2^2$ for all $f \in C([a, b])$.

Some comments before the proof.

1. The second equivalence is called *Parseval's Formula.*

2. A function g is in the linear span of the ψ_n's if there are nonnegative integers $n_0, \ldots, n_k$ and scalars α_i for $0 \leq i \leq k$ with
$$g(x) = \sum_{i=0}^{k} \alpha_i \psi_{n_i}(x).$$
Note that the sum is finite and that, as a matter of definition, $\alpha_i = \langle g, \psi_{n_i} \rangle$.

3. The difficulty is that it is not *a priori* the case that we can choose $\alpha_i = \langle f, \psi_{n_i} \rangle$!

We shall now prove the easy direction of the theorem. The harder direction is beyond the scope of this course.

PROOF: Suppose Parseval's Formula is satisfied for all $f \in C([a,b])$. Then given such an f and $\varepsilon_0 > 0$, there is an $N \in \mathbb{N}$ such that

$$\|f\|_2^2 - \sum_{n=0}^{N} \langle f, \psi_n \rangle^2 < \varepsilon_0^2.$$

Set $g = \sum_{n=0}^{N} \langle f, \psi_n \rangle \psi_n$. Recall from the proof of Bessel's Inequality that

$$\|f - g\|_2^2 = \|f\|_2^2 - \|g\|_2^2 = \|f\|_2^2 - \sum_{n=0}^{N} \langle f, \psi_n \rangle^2 < \varepsilon_0^2.$$

By taking square roots of both sides, we then arrive at

$$\|f - g\|_2 < \varepsilon_0.$$

This ends the proof of the easy direction.

Now we shall show how the first equivalence in the theorem is satisfied for all $f \in C([-c,c])$. In fact, we will do better and show the following:

Theorem: For every $\varepsilon > 0$ and $f \in C([-c,c])$, there is a g in the linear span of the the Fourier system with

$$\|f - g\|_\infty < \varepsilon.$$

We call such a g a *trigonometric polynomial.* The proof is appropriated from Rudin's *Real and Complex Analysis* and is, like most results in that text, pure magic.

PROOF: Define

$$Q_k(x) = c_k \left(\frac{1 + \cos(\frac{\pi x}{c})}{2} \right)^k$$

where $k \in \mathbb{N}$ and

$$c_k = \frac{2c}{\int_{-c}^{c} \left(\frac{1+\cos(\frac{\pi x}{c})}{2}\right)^k \, dx}.$$

In your homework, you will show that $\left(\frac{1+\cos(\frac{\pi x}{c})}{2}\right)^k$ is a trig polynomial. Now by definition, $\frac{1}{2c}\int_{-c}^{c} Q_k(x)\, dx = 1$, as the constant c_k is chosen precisely to make this true. Since $-1 \leq \cos(\frac{\pi x}{c})$, we have that $Q_k(x) \geq 0$ for all $x \in [-c, c]$. Note also $\cos(\frac{\pi x}{c})$ is an even function, and so $Q_k(x)$ is even for all $k \geq 1$. Hence,

$$1 = \frac{c_k}{2c}\int_{-c}^{c} \left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k \, dx = \frac{c_k}{c}\int_{0}^{c} \left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k \, dx$$

Using the fact that $1 \geq \sin\left(\frac{\pi x}{c}\right)$,

$$1 = \frac{c_k}{c}\int_{0}^{c} \left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k \, dx \geq \frac{c_k}{c}\int_{0}^{c} \left(\frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}\right)^k \sin\left(\frac{\pi x}{c}\right) dx.$$

By using the substitution $u = \frac{1+\cos\left(\frac{\pi x}{c}\right)}{2}$, we may evaluate this last integral as

$$\frac{2c_k}{\pi}\int_{0}^{1} u^k \, dx = \frac{2c_k}{\pi}\left(\frac{u^{k+1}}{k+1}\right)\Big|_0^1 = \frac{2c_k}{\pi(k+1)},$$

and so $1 \geq \frac{2c_k}{\pi(k+1)}$, which implies

$$1 \leq \frac{\pi(k+1)}{2c_k} \tag{1}$$

Note the derivative of Q_k is

$$Q_k'(x) = -\frac{\pi k}{c}\frac{Q_{k-1}(x)}{c_{k-1}}\sin\left(\frac{\pi x}{c}\right)$$

and both Q_{k-1} and $\sin\left(\frac{\pi x}{c}\right)$ are positive on $[0, c]$. Therefore, $Q_k' \leq 0$ on $[0, c]$, and so Q_k is decreasing on $[0, c]$.

Since Q_k is decreasing on $[0,c]$, then if $\delta \le x \le c$,

$$\begin{aligned} Q_k(x) \le Q_k(\delta) &= c_k \left(\frac{1+\cos\left(\frac{\pi\delta}{c}\right)}{2} \right)^k \\ &\le \frac{\pi(k+1)}{2c_k} \cdot c_k \left(\frac{1+\cos\left(\frac{\pi\delta}{c}\right)}{2} \right)^k \\ &= \frac{\pi(k+1)}{2} \left(\frac{1+\cos\left(\frac{\pi\delta}{c}\right)}{2} \right)^k . \end{aligned}$$

As $\delta > 0$, we have that $\cos\left(\frac{\pi\delta}{c}\right) < 1$, and so $\frac{1+\cos\left(\frac{\pi\delta}{c}\right)}{2} < 1$. Therefore, regardless of the value of δ, if we set $a = \frac{1+\cos\left(\frac{\pi\delta}{c}\right)}{2}$, we have for all $\delta < x \le c$,

$$\lim_{k\to\infty} Q_k(x) \le \lim_{k\to\infty} Q_k(\delta) = \frac{\pi}{2} \lim_{k\to\infty} \frac{(k+1)}{a^{-k}} = \frac{\pi}{2} \lim_{k\to\infty} \frac{1}{-\ln(a)a^{-k}} = 0 \qquad (2)$$

by l'Hopital's rule. The fact that Q_k is even for all k implies that this result extends to show that

$$\lim_{k\to\infty} Q_k(-x) \le \lim_{k\to\infty} Q_k(\delta) = 0$$

for all $0 < \delta \le x \le c$.

Unfortunately, the Q_k's are not quite good enough to prove what we want since they have no obvious relationship to our given function $f \in C([-c,c])$. For the moment, extend f to all real numbers by setting $h(x+2nc) = f(x)$ for all integers n and $x \in [-c,c)$. This new function may have no resemblance to the original f outside the interval $[-c,c)$, but since we only care about the behavior of f inside the interval, this is justified. We further define, for all real numbers x,

$$g_k(x) = \frac{1}{2c} \int_{-c}^{c} h(x-y) Q_k(y)\ dy.$$

for $k \ge 1$. Then by substituting $u = x - y$,

$$g_k(x) = \frac{1}{2c} \int_{-c}^{c} h(x-y) Q_k(y)\ dy = -\frac{1}{2c} \int_{x+c}^{x-c} h(u) Q_k(x-u)\ du.$$

Now the integrand is a $2c$ periodic function over an interval of length $2c$, and so it follows that

$$\int_{x+c}^{x-c} h(u)Q_k(x-u)\ du = \int_{c}^{-c} h(u)Q_k(x-u)\ du = -\int_{-c}^{c} h(u)Q_k(x-u)\ du.$$

Hence,

$$g_k(x) = -\frac{1}{2c}\int_{x+c}^{x-c} h(u)Q_k(x-u)\ du = \frac{1}{2c}\int_{-c}^{c} h(u)Q_k(x-u)\ du = \frac{1}{2c}\int_{-c}^{c} f(u)Q_k(x-u)\ du$$

by the definition of h.

Recall that the Q_k's are trig polynomials, so for each k, there are non-negative integers $n_{0,k},\ldots,n_{j,k}$ and scalars $\alpha_{0,k},\ldots,\alpha_{j_k,k}$ and $\beta_{1,k},\ldots,\beta_{j_k,k}$ satisfying

$$Q_k(x) = \alpha_{0,k} + \sum_{i=1}^{j_k}\left(\alpha_{i,k}\cos\left(\frac{\pi n_{i,k}x}{c}\right) + \beta_{i,k}\sin\left(\frac{\pi n_{i,k}x}{c}\right)\right).$$

We may assume the same index set for both β's and α's by a judicious use of zeros in the coefficients. Then

$$\begin{aligned}
g_k(x) &= \frac{1}{2c}\int_{-c}^{c} f(u)Q_k(x-u)\ du \\
&= \frac{1}{2c}\int_{-c}^{c} f(u)\alpha_{0,k} + f(u)\sum_{i=1}^{j_k}\left(\alpha_{i,k}\cos\left(\frac{\pi n_{i,k}(x-u)}{c}\right) + \beta_{i,k}\sin\left(\frac{\pi n_{i,k}(x-u)}{c}\right)\right)\ du \\
&= \frac{1}{2c}\int_{-c}^{c} f(u)\alpha_{0,k}\ du \\
&+ \frac{1}{2c}\sum_{i=1}^{j_k}\int_{-c}^{c}\left(\alpha_{i,k}\left(\cos\left(\frac{\pi n_{i,k}x}{c}\right)\cos\left(\frac{\pi n_{i,k}u}{c}\right) + \sin\left(\frac{\pi n_{i,k}x}{c}\right)\sin\left(\frac{\pi n_{i,k}u}{c}\right)\right)\right)\ du \\
&+ \frac{1}{2c}\sum_{i=1}^{j_k}\int_{-c}^{c}\beta_{i,k}\left(\left(\sin\left(\frac{\pi n_{i,k}x}{c}\right)\cos\left(\frac{\pi n_{i,k}u}{c}\right) - \cos\left(\frac{\pi n_{i,k}x}{c}\right)\sin\left(\frac{\pi n_{i,k}u}{c}\right)\right)\right)\ du
\end{aligned}$$

where we have used the identities $\cos(A-B) = \cos(A)\cos(B)+\sin(A)\sin(B)$ and $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$. Rewriting a single integral,

$$\int_{-c}^{c}\alpha_{i,k}\left(\cos\left(\frac{\pi n_{i,k}x}{c}\right)\cos\left(\frac{\pi n_{i,k}u}{c}\right) + \sin\left(\frac{\pi n_{i,k}x}{c}\right)\sin\left(\frac{\pi n_{i,k}u}{c}\right)\right)\ du$$

$$= \alpha_{i,k} \cos\left(\frac{\pi n_{i,k} x}{c}\right) \left(\int_{-c}^{c} \cos\left(\frac{\pi n_{i,k} u}{c}\right)\ du\right) + \alpha_{i,k} \sin\left(\frac{\pi n_{i,k} x}{c}\right) \left(\int_{-c}^{c} \sin\left(\frac{\pi n_{i,k} u}{c}\right)\ du\right).$$

Since each integral with respect to u is just another constant, this shows that any individual integral in the first sum can be written as a trig polynomial, and a similar tedious calculation will show the same for the second sum. We conclude that g_k is a trig polynomial.